

IRREDUCIBLE MODULES FOR EQUIVARIANT MAP SUPERALGEBRAS AND THEIR EXTENSIONS

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ABSTRACT. Let Γ be a group acting on a scheme X and on a Lie superalgebra \mathfrak{g} . The corresponding equivariant map superalgebra $M(\mathfrak{g}, X)^\Gamma$ is the Lie superalgebra of equivariant regular maps from X to \mathfrak{g} . In this paper we complete the classification of finite-dimensional irreducible $M(\mathfrak{g}, X)^\Gamma$ -modules when \mathfrak{g} is a finite-dimensional simple Lie superalgebra, X is of finite type, and Γ is a finite abelian group acting freely on the rational points of X . We also describe extensions between these irreducible modules in terms of extensions between modules for certain finite-dimensional Lie superalgebras. As an application, when Γ is trivial and \mathfrak{g} is of type $B(0, n)$, we describe the block decomposition of the category of finite-dimensional $M(\mathfrak{g}, X)^\Gamma$ -modules in terms of spectral characters for \mathfrak{g} .

1. INTRODUCTION

Lie superalgebras have a wide range of applications in many areas of physics and mathematics, such as supersymmetry, string theory, conformal field theory and number theory. In the study of symmetry, for instance, while Lie algebras describe bosonic degrees of freedom, Lie superalgebras also allow fermionic degrees of freedom [Var04]. In number theory, affine Kac-Moody superalgebras and their representations can be used to study problems related to sums of squares and sums of triangular numbers [KW94]. For more examples, see for instance [FL85, Ser85, GLS01, FK02].

It is usually the case that the representation theory of Lie superalgebras is more complicated than that of their Lie algebra counterparts. For instance, the category of finite-dimensional modules for a finite-dimensional simple Lie algebra is always semisimple, while the category of finite-dimensional modules for a finite-dimensional simple Lie superalgebra is not necessarily so. It is thus important to describe extensions between their irreducible modules. Despite being a subject of intense study since the birth of supersymmetry, these extensions are not known in general. And in contrast with extensions for Lie algebras, their study is often done case by case. See, for instance, [FL84, Fuk86, Pol88, SZ98, SZ99, Gru00, Gru03, BKN10, Bag12] for some of these results.

The main goal of the current paper is to develop the representation theory of certain Lie superalgebras, known as equivariant map superalgebras. Equivariant map superalgebras generalize, on the one hand, simple Lie superalgebras, and on the other hand, current and loop Lie algebras. They are constructed in the following way. Consider a scheme X , a Lie superalgebra \mathfrak{g} , and a group Γ that acts on X and \mathfrak{g} by automorphisms. The corresponding equivariant map superalgebra $M(X, \mathfrak{g})^\Gamma$ is the Lie superalgebra of equivariant regular maps from X to \mathfrak{g} . If one denotes by A the coordinate ring of X , then $M(X, \mathfrak{g})^\Gamma$ can be identified with the Lie subsuperalgebra $(\mathfrak{g} \otimes A)^\Gamma$ of $\mathfrak{g} \otimes A$ consisting of its Γ -fixed points.

Date: April 7, 2016.

Research of the first author was supported by FAPESP grant 2013/08430-4.

Research of the second author was supported by CNPq grant 232462/2014-3.

In the particular case where \mathfrak{g} is a Lie algebra, $M(X, \mathfrak{g})^\Gamma$ is called an equivariant map algebra. The representation theory of these Lie algebras has been a subject of intense research for the last thirty years (see, for instance, the survey [NS13]). One reason is that representations of $M(\mathbb{C}^\times, \mathfrak{g})^\Gamma$, known as twisted loop algebras, and $M(\mathbb{C}, \mathfrak{g})^\Gamma$, known as twisted current algebras, are closely related to those of affine Kac-Moody Lie algebras. In fact, when \mathfrak{g} is a finite-dimensional simple Lie algebra and Γ is a subgroup of the group of automorphisms of the Dynkin diagram of \mathfrak{g} , the twisted current algebra is a parabolic subalgebra of the affine Kac-Moody Lie algebra associated to \mathfrak{g} and Γ , and the twisted loop algebra is its centerless derived subalgebra (see [Kum02, Section 13.1]).

Finite-dimensional irreducible representations of equivariant map algebras were classified by Neher, Savage and Senesi [NSS12] in the case where \mathfrak{g} is a finite-dimensional Lie algebra and Γ is a finite group. Finite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero and finite-dimensional irreducible representations of the so-called basic classical Lie superalgebras were classified by Kac in [Kac77] and [Kac78]. In [Sav14], Savage classified irreducible finite-dimensional representations of equivariant map superalgebras in the case where \mathfrak{g} is a basic classical Lie superalgebra (or $\mathfrak{sl}(n, n)$, $n \geq 1$, if Γ is trivial), X has a finitely generated coordinate ring and Γ is a finite abelian group acting freely on the rational points of X . Moving beyond basic classical Lie superalgebras, the first author, Moura and Savage classified finite-dimensional irreducible representations of equivariant map queer Lie superalgebras in [CMS15]. While in the basic classical setting those irreducible representations were isomorphic to tensor products of generalized evaluation representations, in the queer case they are irreducible products of evaluation representations (see Section 2.3 for details). In [Bag15], Bagci extended this classification to equivariant map superalgebras where \mathfrak{g} is of Cartan type. In the current paper, we complete this classification by describing finite-dimensional irreducible $M(\mathfrak{g}, X)^\Gamma$ -modules when \mathfrak{g} is a periplectic Lie superalgebra.

Equipped with a complete classification of irreducible objects in the category of finite-dimensional $M(\mathfrak{g}, X)^\Gamma$ -modules, one can inquire about their extensions. This is the second problem that we address in the current paper. From now on, let $M(\mathfrak{g}, X)^\Gamma$ be identified with $(\mathfrak{g} \otimes A)^\Gamma$ (as described above). Our main result (Theorem 4.8) reduces the problem of computing extensions between finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$ -modules to that of computing homomorphisms and extensions between finite-dimensional indecomposable modules for a finite-dimensional Lie superalgebra of the form $\mathfrak{g} \otimes A/\mathfrak{m}^n$, where \mathfrak{m} is a maximal ideal of A and n is a positive integer. In fact, even more can be said if \mathfrak{g} is of type II, \mathfrak{p} , \mathfrak{q} , H , S or \tilde{S} . In these cases, $n = 1$, and extensions between finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$ -modules are completely described by homomorphisms and extensions between finite-dimensional indecomposable \mathfrak{g} -modules. Using the formulas given in Theorem 4.8, we are able to describe the block decomposition of the category of finite-dimensional $\mathfrak{g} \otimes A$ -modules in terms of spectral characters when \mathfrak{g} is isomorphic to $\mathfrak{osp}(1, 2n)$, $n \geq 0$ (see Theorem 5.5). This description extends results of Kodera [Kod10, Proposition 4.5] (also compare it with [NS15, Proposition 6.6]).

This paper is organized as follows. In Section 2 we fix the notation and state some results that will be used throughout the paper. In Section 3 we describe the structure of the so-called periplectic Lie superalgebras in order to construct and classify the finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$ -modules when \mathfrak{g} is of periplectic type (see Section 3.3). In Section 4 we develop a general technique to describe extensions between finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$ -modules. In Section 4.1 we prove a general result to compute p -extensions between finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$ -modules (see Proposition 4.1). We then apply this result to the case where the tensor product of two finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$ -modules is completely reducible and, in particular,

when they have disjoint supports (see Corollary 4.2 and Corollary 4.3). In Section 4.2 we prove general results describing cohomologies and 1-extensions for finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$ -modules in terms of homomorphisms and extensions of modules for certain truncated map algebras associated to \mathfrak{g} (see Proposition 4.4 and Theorem 4.8). We also obtain some vanishing conditions for cohomologies and extensions for finite-dimensional $(\mathfrak{g} \otimes A)^\Gamma$ -modules (see Proposition 4.6 and Corollary 4.7). In Section 5 we apply the technique developed in Section 4 and describe the block decomposition of the category of finite-dimensional $\mathfrak{g} \otimes A$ -modules in the case where \mathfrak{g} is isomorphic to $\mathfrak{osp}(1, 2n)$ with $n \geq 0$.

Acknowledgments. The authors would like to thank E. Neher and A. Savage for helpful discussions and for their comments on earlier versions of this paper. The first author would like to thank FAPESP and the second author would like to thank CNPq for their financial supports.

2. NOTATION AND PRELIMINARIES

2.1. Notation. Let \mathbb{C} denote the field of complex numbers and \mathbb{Z}_2 denote the finite field with two elements $\{0, 1\}$. We assume that all our vector spaces are over \mathbb{C} . A vector space V is said to be a super space if it is \mathbb{Z}_2 -graded; that is, there exist subspaces $V_0, V_1 \subseteq V$ such that $V = V_0 \oplus V_1$. We denote by $|\cdot|$ the degree of a homogeneous element in a super space V ; that is, $|v| = z$ for all $v \in V_z$ and $z \in \mathbb{Z}_2$.

2.2. Finite-dimensional Lie superalgebras. In this subsection we follow [Mus12, Gav14]. The following examples will be used in this paper.

Example 2.1. Consider a super space $V = V_0 \oplus V_1$. The associative algebra $\text{End}(V)$, consisting of linear endomorphisms of V admits a \mathbb{Z}_2 -grading defined by

$$\text{End}(V)_z = \{\phi \in \text{End}(V) : \phi(v) \in V_{z+z'} \text{ for all } v \in V_{z'}, z' \in \mathbb{Z}_2\},$$

and a Lie superbracket defined by $[\phi, \psi] = \phi \circ \psi - (-1)^{|\phi||\psi|} \psi \circ \phi$. This Lie superalgebra will be denoted by $\mathfrak{gl}(V)$.

Example 2.2. Consider a super space $V = V_0 \oplus V_1$. Its exterior algebra $\Lambda(V)$ admits a \mathbb{Z}_2 -grading defined by $|v_1 \wedge \cdots \wedge v_n| = |v_1| + \cdots + |v_n|$ for all $v_1, \dots, v_n \in V$, and a Lie superbracket defined by $[v, w] = v \wedge w - (-1)^{|v||w|} w \wedge v$ for all $v, w \in \Lambda(V)$. Denote by $\Lambda^n(V)$ the subspace of $\Lambda(V)$ spanned by $(v_1 \wedge \cdots \wedge v_n)$, with $v_1, \dots, v_n \in V$. Since $v \wedge w = -(-1)^{|v||w|} w \wedge v$ for all homogeneous elements $v, w \in V$, it follows that

$$\Lambda^n(V) = \bigoplus_{i+j=n} \Lambda^i(V_0) \otimes S^j(V_1) \quad \text{for all } n \geq 0.$$

For any Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, the subspace \mathfrak{g}_0 inherits the structure of a Lie algebra and the subspace \mathfrak{g}_1 inherits the structure of a \mathfrak{g}_0 -module. A finite-dimensional simple Lie superalgebra \mathfrak{g} is said to be classical if the \mathfrak{g}_0 -module \mathfrak{g}_1 is completely reducible. Otherwise it is said to be of Cartan type. When \mathfrak{g} is a classical Lie superalgebra, the \mathfrak{g}_0 -module \mathfrak{g}_1 is either irreducible or a direct sum of two irreducible modules. In the first case, \mathfrak{g} is said to be of type II, and in the second case, \mathfrak{g} is said to be of type I. A classical Lie superalgebra is said to be basic if it admits an even nondegenerate invariant bilinear form. Otherwise, it is said to be strange. Table 1 summarizes the classification of finite-dimensional simple Lie superalgebras.

When \mathfrak{g} is a classical Lie superalgebra, a Cartan subalgebra of \mathfrak{g} is defined to be a Cartan subalgebra of the Lie algebra \mathfrak{g}_0 . When \mathfrak{g} is a Lie superalgebra of Cartan type, it admits a \mathbb{Z} -grading

Superalgebra	Also known as	Classification
$A(m, n)$	$\mathfrak{sl}(m+1, n+1)$	Classical, basic, of type I
$A(n, n)$	$\mathfrak{psl}(n+1, n+1)$	Classical, basic, of type I
$B(m, n)$	$\mathfrak{osp}(2m+1, 2n)$	Classical, basic, of type II
$C(n)$	$\mathfrak{osp}(2, 2n-2)$	Classical, basic, of type I
$D(m, n)$	$\mathfrak{osp}(2m, 2n)$	Classical, basic, of type II
$F(4)$	-	Classical, basic, of type II
$G(3)$	-	Classical, basic, of type II
$D(2, 1, \alpha)$	-	Classical, basic, of type II
$\mathfrak{q}(n)$	-	Classical strange
$\mathfrak{p}(n)$	-	Classical strange
$H(n)$	-	Cartan type
$\tilde{S}(n)$	-	Cartan type
$S(n)$	-	Cartan type
$W(n)$	-	Cartan type

TABLE 1. Classification of finite-dimensional simple Lie superalgebras.

$\mathfrak{g} = \bigoplus_{i \geq -1} \mathfrak{g}_i$ that is compatible with the \mathbb{Z}_2 -grading; that is, $\mathfrak{g}_0 = \bigoplus_{i=2k} \mathfrak{g}_i$ and $\mathfrak{g}_1 = \bigoplus_{i=2k+1} \mathfrak{g}_i$, and such that \mathfrak{g}_0 is a reductive Lie algebra. In this case, a Cartan subalgebra of \mathfrak{g} is defined to be a Cartan subalgebra of the Lie algebra \mathfrak{g}_0 . It is worth noting that when \mathfrak{g} is of type \tilde{S} , this is only a \mathbb{Z} -grading as a vector space.

Let \mathfrak{g} be a finite-dimensional simple Lie superalgebra, and let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} . The action of \mathfrak{h} on \mathfrak{g} is diagonalizable and we have a root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha, \quad \text{where} \quad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

Let $\Delta = \{\alpha \in \mathfrak{h}^* : \mathfrak{g}_\alpha \neq 0\}$ denote the set of roots of \mathfrak{g} and Q denote the subgroup of \mathfrak{h}^* generated by Δ .

Let \mathfrak{g} is a finite-dimensional simple Lie superalgebra. Then the set of isomorphism classes of finite-dimensional irreducible \mathfrak{g} -modules is in bijection with a subset Λ^+ of \mathfrak{h}^* (see [Kac77, Theorem 8]). We denote by Λ the subgroup of \mathfrak{h}^* generated by Λ^+ , and by $V(\lambda)$ the irreducible \mathfrak{g} -module corresponding to $\lambda \in \Lambda^+$, namely the unique finite-dimensional irreducible \mathfrak{g} -module of highest weight λ . Let $\lambda^* \in \Lambda^+$ denote the highest weight of the dual \mathfrak{g} -module $V(\lambda)^*$.

The following result will be used in Section 5.

Theorem 2.3 ([DH76, Theorem 4.1]). *Any finite-dimensional module for $B(0, n) = \mathfrak{osp}(1, 2n)$, $n > 0$, is completely reducible.*

Let \mathfrak{g}^1 and \mathfrak{g}^2 be two finite-dimensional Lie superalgebras, and V_1 and V_2 be irreducible finite-dimensional modules for \mathfrak{g}^1 and \mathfrak{g}^2 respectively. The $(\mathfrak{g}^1 \oplus \mathfrak{g}^2)$ -module $V_1 \otimes V_2$ is reducible only if $\text{End}_{\mathfrak{g}^1}(V_1)_{\bar{1}} \neq 0$ and $\text{End}_{\mathfrak{g}^2}(V_2)_{\bar{1}} \neq 0$ (see [Che95, Proposition 8.4]). In this case, by Schur's Lemma for Lie superalgebras, $\text{End}_{\mathfrak{g}^i}(V_i)_{\bar{1}} = \mathbb{C}\varphi_i$ for some $\varphi_i^2 = -1$, $i = 1, 2$, and

$$\hat{V} = \{v \in V_1 \otimes V_2 : (\sqrt{-1}\varphi_1 \otimes \varphi_2)v = v\}$$

is an irreducible $(\mathfrak{g}^1 \oplus \mathfrak{g}^2)$ -submodule satisfying $V_1 \otimes V_2 \cong \widehat{V} \oplus \widehat{V}$ (see [Che95, p. 27]). Define the irreducible product $V_1 \widehat{\otimes} V_2$ to be

$$V_1 \widehat{\otimes} V_2 = \begin{cases} V_1 \otimes V_2, & \text{if } V_1 \otimes V_2 \text{ is irreducible;} \\ \widehat{V}, & \text{otherwise.} \end{cases}$$

If \mathfrak{g}^1 and \mathfrak{g}^2 are finite-dimensional simple Lie superalgebras not of type \mathfrak{q} , then the irreducible product is equal to the tensor product.

Given $\ell > 1$, finite-dimensional simple Lie superalgebras $\mathfrak{g}^1, \dots, \mathfrak{g}^\ell$, and irreducible finite-dimensional \mathfrak{g}^i -modules V_i for all $i = 1, \dots, \ell$, define the $(\mathfrak{g}^1 \oplus \dots \oplus \mathfrak{g}^\ell)$ -module $V_1 \widehat{\otimes} \dots \widehat{\otimes} V_\ell$ to be

$$V_1 \widehat{\otimes} \dots \widehat{\otimes} V_\ell = (V_1 \widehat{\otimes} \dots \widehat{\otimes} V_{\ell-1}) \widehat{\otimes} V_\ell.$$

Up to isomorphism, $\widehat{\otimes}$ is associative and commutative (see [CMS15, Lemma 6.2]). Also, define

$$\kappa(V_1, \dots, V_\ell) = \sum_{i=2}^{\ell} \dim \text{End}_{(\mathfrak{g}^1 \oplus \dots \oplus \mathfrak{g}^{i-1})} (V_1 \widehat{\otimes} \dots \widehat{\otimes} V_{i-1})_{\bar{1}} \dim \text{End}_{\mathfrak{g}^i} (V_i)_{\bar{1}}.$$

One can prove by induction that $\kappa(V_1, \dots, V_\ell) \leq \ell - 1$ and that

$$V_1 \otimes \dots \otimes V_\ell \cong (V_1 \widehat{\otimes} \dots \widehat{\otimes} V_\ell)^{\oplus 2^k} \quad \text{for } k = \kappa(V_1, \dots, V_\ell).$$

2.3. Equivariant map superalgebras. Let A be an associative commutative \mathbb{C} -algebra with unit, and \mathfrak{g} be a Lie superalgebra. The map superalgebra $\mathfrak{g} \otimes A$ is the Lie superalgebra with underlying vector space $\mathfrak{g} \otimes_{\mathbb{C}} A$, with \mathbb{Z}_2 -grading given by $(\mathfrak{g} \otimes A)_z = \mathfrak{g}_z \otimes A$, $z \in \mathbb{Z}_2$, and with Lie superbracket extending bilinearly

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab, \quad \text{for all } x, y \in \mathfrak{g} \text{ and } a, b \in A.$$

Let Γ be a group acting on \mathfrak{g} and A by automorphisms. One can induce an action of Γ on $\mathfrak{g} \otimes A$ by extending by linearity

$$\gamma(x \otimes a) = \gamma(x) \otimes \gamma(a)$$

for all $\gamma \in \Gamma$, $x \in \mathfrak{g}$ and $a \in A$. The equivariant map superalgebra $(\mathfrak{g} \otimes A)^\Gamma$ is the Lie subsuperalgebra of $\mathfrak{g} \otimes A$ consisting of its Γ -fixed points:

$$(\mathfrak{g} \otimes A)^\Gamma = \{x \in \mathfrak{g} \otimes A \mid \gamma x = x \text{ for all } \gamma \in \Gamma\}.$$

Let $\text{MaxSpec}(A)$ denote the set of maximal ideals of A . Notice that the action of Γ on A induces an action of Γ on $\text{MaxSpec}(A)$ explicitly given by $\gamma \mathfrak{m} = \{\gamma a \mid a \in \mathfrak{m}\} \in \text{MaxSpec}(A)$, for all $\mathfrak{m} \in \text{MaxSpec}(A)$ and $\gamma \in \Gamma$. If \mathfrak{g} is a finite-dimensional simple Lie superalgebra, A is finitely generated, and Γ is a finite abelian group acting freely on $\text{MaxSpec}(A)$, then every finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$ -module can be described in terms of generalized evaluation modules. These generalized evaluation modules are defined in the following way. Given $\mathfrak{m} \in \text{MaxSpec}(A)$ and $n > 0$, define $\text{ev}_{\mathfrak{m}^n}$ to be the homomorphism of Lie superalgebras given by the composition

$$\text{ev}_{\mathfrak{m}^n}: \mathfrak{g} \otimes A \rightarrow \mathfrak{g} \otimes A / \mathfrak{g} \otimes \mathfrak{m}^n \xrightarrow{\cong} \mathfrak{g} \otimes A / \mathfrak{m}^n.$$

Since Γ acts freely on $\text{MaxSpec}(A)$, the restriction of $\text{ev}_{\mathfrak{m}^n}$ to $(\mathfrak{g} \otimes A)^\Gamma$ is surjective (see [Sav14, Lemma 5.6]), and induces a homomorphism of Lie superalgebras

$$\text{ev}_{\mathfrak{m}^n}^\Gamma: (\mathfrak{g} \otimes A)^\Gamma \rightarrow \mathfrak{g} \otimes A / \mathfrak{m}^n.$$

Given a $\mathfrak{g} \otimes A / \mathfrak{m}^n$ -module V with associated representation $\rho: \mathfrak{g} \otimes A / \mathfrak{m}^n \rightarrow \mathfrak{gl}(V)$, define $\text{ev}_{\mathfrak{m}^n}^{\Gamma*}(V)$ to be the $(\mathfrak{g} \otimes A)^\Gamma$ -module with associated representation given by the pull-back of ρ along $\text{ev}_{\mathfrak{m}^n}^\Gamma$,

$$\text{ev}_{\mathfrak{m}^n}^{\Gamma*}(\rho): (\mathfrak{g} \otimes A)^\Gamma \xrightarrow{\text{ev}_{\mathfrak{m}^n}^\Gamma} \mathfrak{g} \otimes A / \mathfrak{m}^n \xrightarrow{\rho} \mathfrak{gl}(V).$$

Given $\mathfrak{m} \in \text{MaxSpec}(A)$ and $n > 0$, denote by $\text{Irred}(\mathfrak{g} \otimes A/\mathfrak{m}^n)$ the set of isomorphism classes of finite-dimensional irreducible $\mathfrak{g} \otimes A/\mathfrak{m}^n$ -modules, and denote by \mathcal{R} the disjoint union

$$\mathcal{R} = \bigsqcup_{\substack{n > 0 \\ \mathfrak{m} \in \text{MaxSpec}(A)}} \text{Irred}(\mathfrak{g} \otimes A/\mathfrak{m}^n).$$

Notice that the action of Γ on $\mathfrak{g} \otimes A$ induces an action of Γ on \mathcal{R} . Explicitly, let $[V]$ be an element in \mathcal{R} , and let V be a representative of the class $[V]$ with associated representation $\rho: \mathfrak{g} \otimes A/\mathfrak{m}^n \rightarrow \mathfrak{gl}(V)$. For each $\gamma \in \Gamma$, define $\gamma \cdot [V]$ in \mathcal{R} to be the isomorphism class of the $\mathfrak{g} \otimes A/(\gamma\mathfrak{m})^n$ -module V^γ , whose associated representation $\rho': \mathfrak{g} \otimes A/(\gamma\mathfrak{m})^n \rightarrow \mathfrak{gl}(V)$ is given by $\rho'(x) = \rho(\gamma^{-1}x)$, for all $x \in \mathfrak{g} \otimes A/(\gamma\mathfrak{m})^n$.

Denote by \mathcal{P} the set of Γ -equivariant functions $\pi: \text{MaxSpec}(A) \rightarrow \mathcal{R}$ satisfying the following conditions:

- For each $\mathfrak{m} \in \text{MaxSpec}(A)$, $\pi(\mathfrak{m}) \in \text{Irred}(\mathfrak{g} \otimes A/\mathfrak{m}^n)$ for some $n > 0$;
- $\pi(\mathfrak{m})$ is the isomorphism class of the trivial module for all but finitely many $\mathfrak{m} \in \text{MaxSpec}(A)$.

Notice that for any two representatives V and V' of $\pi(\mathfrak{m}) \in \text{Irred}(\mathfrak{g} \otimes A/\mathfrak{m}^n)$, there is an isomorphism of $(\mathfrak{g} \otimes A)^\Gamma$ -modules $\text{ev}_{\mathfrak{m}^n}^\Gamma V \cong \text{ev}_{\mathfrak{m}^n}^\Gamma V'$. Thus we will abuse notation and for each maximal ideal \mathfrak{m} of A , we will denote by $\pi(\mathfrak{m})$ an arbitrary but fixed choice of $(\mathfrak{g} \otimes A/\mathfrak{m}^n)$ -module representative of $\pi(\mathfrak{m})$.

Given $\pi \in \mathcal{P}$, define its support to be $\text{Supp}(\pi) = \{\mathfrak{m} \in \text{MaxSpec}(A) \mid \pi(\mathfrak{m}) \text{ is nontrivial}\}$, and let $\text{Supp}_*(\pi)$ be a subset of $\text{MaxSpec}(A)$ which: contains one element of each Γ -orbit in $\text{Supp}(\pi)$ and such that $\mathfrak{m}' \notin \Gamma\mathfrak{m}$ if $\mathfrak{m}, \mathfrak{m}' \in \text{Supp}_*(\pi)$ are distinct. Since every $\pi \in \mathcal{P}$ is Γ -equivariant, up to isomorphism, the $(\mathfrak{g} \otimes A)^\Gamma$ -module $\widehat{\bigotimes_{\mathfrak{m} \in \text{Supp}_*(\pi)} \text{ev}_{\mathfrak{m}^n}^\Gamma \pi(\mathfrak{m})}$ is independent of the choice of $\text{Supp}_*(\pi)$ (see [Sav14, Lemma 5.9]). Thus for every $\pi \in \mathcal{P}$, we fix an arbitrary subset $\text{Supp}_*(\pi)$ as above and define $\mathcal{V}(\pi)$ to be the $(\mathfrak{g} \otimes A)^\Gamma$ -module

$$\mathcal{V}(\pi) = \widehat{\bigotimes_{\mathfrak{m} \in \text{Supp}_*(\pi)} \text{ev}_{\mathfrak{m}^n}^\Gamma \pi(\mathfrak{m})}.$$

Since \mathfrak{g} is a finite-dimensional simple Lie superalgebra, A is an associative, commutative, finitely generated algebra with unit, Γ is a finite abelian group acting on \mathfrak{g} and A by automorphism, and such that its action on $\text{MaxSpec}(A)$ is free, then every finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$ -module is isomorphic to $\mathcal{V}(\pi)$ for a unique $\pi \in \mathcal{P}$. This was proved by Savage when \mathfrak{g} is a basic Lie superalgebra (see [Sav14, §7]), by Bagci when \mathfrak{g} is a Lie superalgebra of Cartan type (see [Bag15, Theorem 4.3]), and by the first author, Moura and Savage when \mathfrak{g} is a queer Lie superalgebra (see [CMS15, Theorem 7.1]). In Section 3, we will show that, if \mathfrak{g} is of type \mathfrak{p} , then every finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$ -module is also isomorphic to $\mathcal{V}(\pi)$ for a unique $\pi \in \mathcal{P}$, and that $n = 1$ for all \mathfrak{m} . This completes the classification of finite-dimensional irreducible modules for equivariant map superalgebras associated to finite-dimensional simple Lie algebras.

Moreover, if \mathfrak{g} is a finite-dimensional simple Lie superalgebra of type II, \mathfrak{q} , H , S or \tilde{S} , then every finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$ -module is isomorphic to $\mathcal{V}(\pi)$ for a unique $\pi \in \mathcal{P}$ with $n = 1$ for all \mathfrak{m} (these modules are called evaluation modules by Savage, see [Sav14, Definition 5.2]). It is important to remark that if \mathfrak{g} is of type I, there exist finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$ -modules which are isomorphic to generalized evaluation modules but not to evaluation modules [ER13, §4].

Remark 2.4. Let $n, n' \geq 0$ with $n' \leq n$, let \mathfrak{m} be a maximal ideal in A , and let V be a $\mathfrak{g} \otimes A/\mathfrak{m}^{n'}$ -module with corresponding representation $\rho: \mathfrak{g} \otimes A/\mathfrak{m}^{n'} \rightarrow \mathfrak{gl}(V)$. Since $n' \leq n$, then $\mathfrak{m}^n \subseteq \mathfrak{m}^{n'}$, and therefore we have a canonical projection $\pi: \mathfrak{g} \otimes A/\mathfrak{m}^n \twoheadrightarrow \mathfrak{g} \otimes A/\mathfrak{m}^{n'}$. We can thus regard V as a $\mathfrak{g} \otimes A/\mathfrak{m}^n$ -module with representation given by $\rho \circ \pi$.

Notice that as representations of $(\mathfrak{g} \otimes A)^\Gamma$, $\text{ev}_{\mathfrak{m}^{n'}}^{\Gamma*}(\rho \circ \pi)$ and $\text{ev}_{\mathfrak{m}^n}^{\Gamma*}(\rho)$ are the same. Hence, the $(\mathfrak{g} \otimes A)^\Gamma$ -modules $\text{ev}_{\mathfrak{m}^{n'}}^{\Gamma*}V$ and $\text{ev}_{\mathfrak{m}^n}^{\Gamma*}V$ are isomorphic. Moreover, if ρ is an irreducible representation of $\mathfrak{g} \otimes A/\mathfrak{m}^n$, then $\rho \circ \pi$ is an irreducible representation of $\mathfrak{g} \otimes A/\mathfrak{m}^{n'}$. As a consequence, given any two finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$ -modules V and V' , one loses no generality in assuming that there exist integers $\ell, n > 0$, maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_\ell \in \text{MaxSpec}(A)$ in distinct Γ -orbits, and for each $i = 1, \dots, \ell$, irreducible $\mathfrak{g} \otimes A/\mathfrak{m}_i^n$ -modules V_i, V'_i , such that $V \cong \widehat{\bigotimes}_{i=1}^\ell \text{ev}_{\mathfrak{m}_i^n}^{\Gamma*} V_i$ and $V' \cong \widehat{\bigotimes}_{i=1}^\ell \text{ev}_{\mathfrak{m}_i^n}^{\Gamma*} V'_i$.

Given $\pi \in \mathcal{P}$, let ℓ, n_1, \dots, n_ℓ be positive integers, let $\mathfrak{m}_1, \dots, \mathfrak{m}_\ell$ be maximal ideals of A in distinct Γ -orbits, and for each $i = 1, \dots, \ell$, let V_i be an irreducible $\mathfrak{g} \otimes A/\mathfrak{m}_i^{n_i}$ -module such that $\mathcal{V}(\pi) \cong \widehat{\bigotimes}_{i=1}^\ell \text{ev}_{\mathfrak{m}_i^{n_i}}^{\Gamma*} V_i$. Define $\kappa(\pi)$ to be

$$(2.3.1) \quad \kappa(\pi) = \kappa(V_1, \dots, V_\ell),$$

and notice that $\mathcal{V}(\pi)^{\oplus 2^{\kappa(\pi)}} \cong \bigotimes_{\mathfrak{m} \in \text{Supp}(\pi)} \text{ev}_{\mathfrak{m}^n}^{\Gamma*} \pi(\mathfrak{m})$ for all $\pi \in \mathcal{P}$.

2.4. Ideals. Let A be an associative commutative \mathbb{C} -algebra with unit. Define the support of an ideal I of A to be

$$\text{Supp}(I) = \{\mathfrak{m} \in \text{MaxSpec } A \mid I \subseteq \mathfrak{m}\}.$$

Lemma 2.5. *Let I and J be ideals of A .*

- (a) *For any $n > 0$, we have $\text{Supp}(I) = \text{Supp}(I^n)$.*
- (b) *If A is finitely generated, then $\text{Supp}(I)$ is finite if and only if I has finite codimension in A .*
- (c) *If $\text{Supp}(I) \cap \text{Supp}(J) = \emptyset$, then $I + J = A$ and $IJ = I \cap J$. Moreover, $I^m + J^n = A$ and $I^m J^n = I^m \cap J^n$ for any $m, n > 0$.*
- (d) *If A is Noetherian, then every ideal $I \subseteq A$ contains a power of its radical. In particular, $\text{rad } I = \prod_{\mathfrak{m} \in \text{Supp}(I)} \mathfrak{m}$.*

Proof. To prove part (a), fix $n > 0$. It is clear that $\text{Supp}(I) \subseteq \text{Supp}(I^n)$. The reverse inclusion follows from the fact that maximal ideals are also prime. Indeed, suppose \mathfrak{m} is a maximal ideal containing I^n . Then for all $a \in I$, we have $a^n \in \mathfrak{m}$. Since maximal ideals are prime ideals, this implies that $a \in \mathfrak{m}$, showing that $I \subseteq \mathfrak{m}$. The proofs of parts (b), (c) and (d) can be found in [Sav14, §2.1]. \square

Lemma 2.6. *Let \mathfrak{g} be a finite-dimensional simple Lie superalgebra. Then every ideal of $\mathfrak{g} \otimes A$ is of the form $\mathfrak{g} \otimes I$, where I is an ideal of A .*

Proof. This is a particular case of [Sav14, Proposition 8.1]. \square

Proposition 2.7. *Let M be a finite-dimensional $\mathfrak{g} \otimes A$ -module. Then there exist integers $n, \ell > 0$ and maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_\ell \subseteq A$, such that $(\mathfrak{g} \otimes \mathfrak{m}_1^n \cdots \mathfrak{m}_\ell^n)M = 0$.*

Proof. Let $\rho: \mathfrak{g} \otimes A \rightarrow \mathfrak{gl}(M)$ be the representation of $\mathfrak{g} \otimes A$ corresponding to M . Notice that $\ker \rho$ is a finite-codimensional ideal of $\mathfrak{g} \otimes A$, since M is finite-dimensional. Thus, by Lemma 2.6, it must be of the form $\mathfrak{g} \otimes I$, for some finite-codimensional ideal I of A . Now, by Lemma 2.5 (b), the fact

that I is finite-codimensional implies that I has finite support. Thus there exist $\ell > 0$ and maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_\ell \subset A$ such that $I \subset \mathfrak{m}_1 \cdots \mathfrak{m}_\ell$. Since these maximal ideals are pairwise distinct, the radical of I is given by $\text{rad } I = \mathfrak{m}_1 \cdots \mathfrak{m}_\ell$. Moreover, since A is assumed to be finitely generated, by Lemma 2.5 (d), there exists $n > 0$, such that $(\text{rad } I)^n \subset I$; that is, such that $\mathfrak{m}_1^n \cdots \mathfrak{m}_\ell^n \subset I$. We thus conclude that there exist integers $n, \ell > 0$ and maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_\ell \subset A$ that satisfy the desired condition. \square

Proposition 2.8. *Let M be a finite-dimensional $(\mathfrak{g} \otimes A)^\Gamma$ -module. Then there exist integers $\ell, n > 0$ and maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_\ell \subseteq A$, such that $\mathfrak{m}_1^n \cdots \mathfrak{m}_\ell^n$ is a Γ -invariant finite-codimensional ideal and $(\mathfrak{g} \otimes \mathfrak{m}_1^n \cdots \mathfrak{m}_\ell^n)^\Gamma M = 0$.*

Proof. By [Sav14, Proposition 8.5], M is the restriction of a finite-dimensional $\mathfrak{g} \otimes A$ -module M' . Since M' is finite-dimensional, by Proposition 2.7, there exist integers $n, k > 0$ and maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_k \subseteq A$, such that $(\mathfrak{g} \otimes \mathfrak{m}_1^n \cdots \mathfrak{m}_k^n) M' = 0$. Consider all ideals of the form $\gamma \mathfrak{m}_i$, with $\gamma \in \Gamma$, $i = 1, \dots, k$. Since Γ is a finite group, we can enumerate the elements of $\{\gamma \mathfrak{m}_i \mid \gamma \in \Gamma, i = 1, \dots, k\}$ as $\mathfrak{m}_1, \dots, \mathfrak{m}_\ell$, with $\ell \leq |\Gamma|k$ and the equality holding only if $\mathfrak{m}_i \notin \Gamma \mathfrak{m}_j$, for $1 \leq i \neq j \leq k$. Notice that $\mathfrak{m}_1^n \cdots \mathfrak{m}_\ell^n$ is a Γ -invariant finite-codimensional ideal and that

$$(\mathfrak{g} \otimes \mathfrak{m}_1^n \cdots \mathfrak{m}_\ell^n)^\Gamma M = (\mathfrak{g} \otimes \mathfrak{m}_1^n \cdots \mathfrak{m}_\ell^n)^\Gamma M' \subseteq (\mathfrak{g} \otimes \mathfrak{m}_1^n \cdots \mathfrak{m}_k^n) M' = 0. \quad \square$$

2.5. Homology and cohomology. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra and V be a \mathfrak{g} -module. Since \mathfrak{g} and V are \mathbb{Z}_2 -graded, $\Lambda^n \mathfrak{g} \otimes V$ and $\text{Hom}_{\mathbb{C}}(\Lambda^n \mathfrak{g}, V)$ are naturally \mathbb{Z}_2 -graded. Namely

$$(\Lambda^n \mathfrak{g} \otimes V)_z = \{x_1 \wedge \cdots \wedge x_n \wedge v \mid |x_1| + \cdots + |x_n| + |v| = z\} \quad \text{and} \\ \text{Hom}_{\mathbb{C}}(\Lambda^n \mathfrak{g}, V)_z = \{f: \Lambda^n \mathfrak{g} \rightarrow V \mid f(\lambda) \in V_{w+z}, \lambda \in (\Lambda^n \mathfrak{g})_w, w \in \mathbb{Z}_2\},$$

for all $n \geq 0$ and $z \in \mathbb{Z}_2$.

Define the homology of \mathfrak{g} with coefficients in V , $H_\bullet(\mathfrak{g}, V)$, to be the homology of the complex

$$(2.5.1) \quad \cdots \rightarrow \Lambda^3 \mathfrak{g} \otimes V \xrightarrow{d_3} \Lambda^2 \mathfrak{g} \otimes V \xrightarrow{d_2} \mathfrak{g} \otimes V \xrightarrow{d_1} V \rightarrow 0,$$

with differentials extending linearly

$$(2.5.2) \quad \begin{aligned} d_n(x_1 \wedge x_2 \wedge \cdots \wedge x_n \otimes v) &= \sum_{i=1}^n (-1)^{\epsilon_i} x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_n \otimes x_i v \\ &+ \sum_{j < k} (-1)^{\epsilon_{j,k}} [x_j, x_k] \wedge x_1 \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge \widehat{x_k} \wedge \cdots \wedge x_n \otimes v, \end{aligned}$$

where $\epsilon_i = i + |x_i|(|x_{i+1}| + \cdots + |x_n|)$, $\epsilon_{j,k} = j + k + \eta_j + \eta_k + |x_j||x_k|$, $\eta_i = |x_i|(|x_1| + \cdots + |x_{i-1}|)$, $x_i \in \mathfrak{g}$ are homogeneous for all $i = 1, \dots, n$, $v \in V$, and the symbol $\widehat{x_i}$ means that we are omitting the term x_i from the wedge product. Observe that the differentials d_\bullet respect the \mathbb{Z}_2 -grading on $\Lambda^\bullet \mathfrak{g} \otimes V$, thus inducing a \mathbb{Z}_2 -grading on $H_\bullet(\mathfrak{g}, V)$.

Given two \mathfrak{g} -modules V and U , define $\text{Ext}_{\mathfrak{g}}^\bullet(V, U)$ to be the cohomology of the cocomplex

$$(2.5.3) \quad 0 \rightarrow \text{Hom}_{\mathbb{C}}(V, U) \xrightarrow{d^0} \text{Hom}_{\mathbb{C}}(\mathfrak{g} \otimes V, U) \xrightarrow{d^1} \text{Hom}_{\mathbb{C}}(\Lambda^2 \mathfrak{g} \otimes V, U) \xrightarrow{d^2} \cdots,$$

where, for all $f \in \text{Hom}_{\mathbb{C}}(\Lambda^n \mathfrak{g} \otimes V, U)$, its image under the differential d^n extends linearly

$$(2.5.4) \quad \begin{aligned} d^n f(x_0 \wedge x_1 \wedge \cdots \wedge x_n \otimes v) &= \sum_{i=1}^n (-1)^{\epsilon_i} x_i f(x_0 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_n \otimes v) \\ &+ \sum_{j < k} (-1)^{\epsilon_{j,k}} f([x_j, x_k] \wedge x_0 \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge \widehat{x_k} \wedge \cdots \wedge x_n \otimes v) \end{aligned}$$

for all homogeneous elements $x_0, \dots, x_n \in \mathfrak{g}$, $n \geq 0$, and $v \in V$. In particular, define $H^\bullet(\mathfrak{g}, U)$, the cohomology of \mathfrak{g} with coefficients in U , to be $\text{Ext}_{\mathfrak{g}}^\bullet(\mathbb{C}, U)$. Notice that, since the differentials d_\bullet respect the \mathbb{Z}_2 -grading on $\Lambda^\bullet \mathfrak{g} \otimes V$, the differentials d^\bullet respect the \mathbb{Z}_2 -grading on $\text{Hom}_{\mathbb{C}}(\Lambda^\bullet \mathfrak{g} \otimes V, U)$, thus inducing a \mathbb{Z}_2 -grading on $\text{Ext}_{\mathfrak{g}}^\bullet(V, U)$. We denote $H^\bullet(\mathfrak{g}, U)_z$ by $H_z^\bullet(\mathfrak{g}, U)$ for all $z \in \mathbb{Z}_2$.

2.6. Homological techniques. We now state some homological techniques that will be used in this paper. This first lemma reduces the computation of Ext between finite-dimensional modules to the computation of a certain cohomology. A proof can be obtained by modifying [Kum02, Lemma 3.1.13].

Lemma 2.9. *If \mathfrak{g} is a Lie superalgebra and U, V and W are finite-dimensional \mathfrak{g} -modules, then we have the following isomorphisms of \mathfrak{g} -modules:*

$$\begin{aligned} U \otimes V &\cong V \otimes U, \quad (U \otimes V)^* \cong U^* \otimes V^*, \\ \text{Hom}_{\mathbb{C}}(U \otimes V, W) &\cong \text{Hom}_{\mathbb{C}}(U, V^* \otimes W), \text{ and} \\ \text{Ext}_{\mathfrak{g}}^n(V, U) &\cong H^n(\mathfrak{g}, V^* \otimes U), \text{ for all } n > 0. \end{aligned}$$

The following lemma reduces the computation of the cohomology of any trivial \mathfrak{g} -module; that is, any module V such that $\mathfrak{g} \cdot V = 0$, to that of the trivial module \mathbb{C} . Its proof follows directly from the definition of the differentials (2.5.2).

Lemma 2.10. *If \mathfrak{g} is a Lie superalgebra and V is a trivial \mathfrak{g} -module, then*

$$H^\bullet(\mathfrak{g}, V) \cong H^\bullet(\mathfrak{g}, \mathbb{C}) \otimes V.$$

The following proposition is a special case of the well known Künneth formula. A proof can be obtained by using universal enveloping algebras and modifying [Wei94, Theorem 3.6.3].

Proposition 2.11. *If $\mathfrak{g}^1, \mathfrak{g}^2$ are Lie superalgebras, U_1, V_1 are \mathfrak{g}^1 -modules and U_2, V_2 are \mathfrak{g}^2 -modules, then*

$$\text{Ext}_{\mathfrak{g}^1 \oplus \mathfrak{g}^2}^n(U_1 \otimes U_2, V_1 \otimes V_2) \cong \bigoplus_{p+q=n} \text{Ext}_{\mathfrak{g}^1}^p(U_1, V_1) \otimes \text{Ext}_{\mathfrak{g}^2}^q(U_2, V_2), \quad n \geq 0.$$

The following theorem is a graded analog of the Lyndon-Hochschild-Serre spectral sequence. In the superalgebra setting, a filtration by powers of an ideal turns out to be \mathbb{Z}_2 -graded, thus yielding two spectral sequences, which converge respectively to even and odd cohomologies. A proof can be found in [Fuk86, Chapter 1, §6.5].

Proposition 2.12. *If \mathfrak{g} is a Lie superalgebra, V is a \mathfrak{g} -module and $\mathfrak{i} \subseteq \mathfrak{g}$ is an ideal, then there exist first-quadrant cohomology convergent spectral sequences*

$$E_2^{p,q} \cong H_z^p(\mathfrak{g}/\mathfrak{i}, H_z^q(\mathfrak{i}, V)) \Rightarrow H_z^{p+q}(\mathfrak{g}, V), \quad z \in \mathbb{Z}_2.$$

The following result is a superalgebra generalization of a well-known result for Lie algebra cohomology.

Lemma 2.13. *For any Lie superalgebra \mathfrak{a} , we have*

$$H_0^1(\mathfrak{a}, \mathbb{C}) \cong (\mathfrak{a}_0/([\mathfrak{a}_0, \mathfrak{a}_0] + [\mathfrak{a}_1, \mathfrak{a}_1]))^* \quad \text{and} \quad H_1^1(\mathfrak{a}, \mathbb{C}) \cong H^0(\mathfrak{a}_0, \mathfrak{a}_1^*) \cong (\mathfrak{a}_1/[\mathfrak{a}_0, \mathfrak{a}_1])^*.$$

Proof. Recall from Section 2.5 that the cocomplex $\Lambda^\bullet \mathfrak{a}^*$ is \mathbb{Z}_2 -graded and that the differential d^\bullet preserves this grading, inducing a \mathbb{Z}_2 -grading on $H^\bullet(\mathfrak{a}, \mathbb{C})$. We will compute each graded component of $H^1(\mathfrak{a}, \mathbb{C})$. First notice that the restriction of d^1 to the even part is $d_0^1: \mathfrak{a}_0^* \rightarrow \Lambda^2 \mathfrak{a}_0^* \oplus S^2 \mathfrak{a}_1^*$, and that the restriction of d^1 to the odd part is $d_1^1: \mathfrak{a}_1^* \rightarrow \mathfrak{a}_0^* \otimes \mathfrak{a}_1^*$.

By definition, $H^1(\mathfrak{a}, \mathbb{C}) = \ker(d^1)/\text{im}(d^0)$, where $d^0: \mathbb{C} \rightarrow \mathfrak{a}^*$ is given by $d^0(\lambda)(x) = x \cdot \lambda = 0$ for all $\lambda \in \mathbb{C}, x \in \mathfrak{a}$, and $d^1: \mathfrak{a}^* \rightarrow \Lambda^2 \mathfrak{a}^*$ is given by $d^1(\varphi)(x \wedge y) = -\varphi([x, y])$ for all $\varphi \in \mathfrak{a}^*, x, y \in \mathfrak{a}$. So, in order to compute $H^1(\mathfrak{a}, \mathbb{C})$ it is enough to determine $\ker(d^1)$.

From the formula of d^1 , it follows that $d_0^1(\varphi) = 0$ if and only if $\varphi([\mathfrak{a}_0, \mathfrak{a}_0] + [\mathfrak{a}_1, \mathfrak{a}_1]) = 0$. Thus $H_0^1(\mathfrak{a}, \mathbb{C}) \cong (\mathfrak{a}_0/[\mathfrak{a}_0, \mathfrak{a}_0] + [\mathfrak{a}_1, \mathfrak{a}_1])^*$. Also from the formula of d^1 , one can see that $\ker(d_1^1)$ is the kernel of d^0 in the cocomplex for computing the cohomology of the Lie algebra \mathfrak{a}_0 with coefficients in \mathfrak{a}_1^* (see Section 2.5). Thus $H_1^1(\mathfrak{a}, \mathbb{C}) \cong H^0(\mathfrak{a}_0, \mathfrak{a}_1^*)$. \square

The next result follows from the previous one.

Lemma 2.14. *Let I be a Γ -invariant ideal of A . If \mathfrak{g} is a finite-dimensional simple Lie superalgebra and \mathfrak{g}_1 is nonzero, then*

$$H_0^1((\mathfrak{g} \otimes I)^\Gamma, \mathbb{C}) \cong \left((\mathfrak{g}_0 \otimes I/I^2)^\Gamma \right)^* \quad \text{and} \quad H_1^1((\mathfrak{g} \otimes I)^\Gamma, \mathbb{C}) \cong \left((\mathfrak{g}_1 \otimes I/I^2)^\Gamma \right)^*.$$

Proof. First notice that $\mathfrak{g}_0 \oplus [\mathfrak{g}_0, \mathfrak{g}_1]$ and $[\mathfrak{g}_1, \mathfrak{g}_1] \oplus \mathfrak{g}_1$ are ideals of \mathfrak{g} . When \mathfrak{g} is simple and \mathfrak{g}_1 is nonzero, this implies that $[\mathfrak{g}_0, \mathfrak{g}_1] = \mathfrak{g}_1$ and $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_0$. The result follows now from Lemma 2.13. \square

The following corollary follows from Lemma 2.14 and the fact that, if A is an associative, commutative algebra with unit, then $A^2 = A$.

Corollary 2.15. *If \mathfrak{g} is a finite-dimensional simple Lie superalgebra with $\mathfrak{g}_1 \neq 0$, A is an associative, commutative algebra with unit, Γ is an abelian group acting by automorphisms on \mathfrak{g} and A , and the action of Γ on $\text{MaxSpec}(A)$ is free, then $H^1((\mathfrak{g} \otimes A)^\Gamma, \mathbb{C}) = 0$.* \square

As a consequence of Corollary 2.15, we see that there are no nontrivial extensions between trivial $(\mathfrak{g} \otimes A)^\Gamma$ -modules M and N , when \mathfrak{g} is a finite-dimensional simple Lie superalgebra with \mathfrak{g}_1 nonzero.

3. EQUIVARIANT PERIPLECTIC MAP LIE SUPERALGEBRAS

3.1. Structure of periplectic Lie superalgebras. Given $n \geq 2$, the periplectic Lie superalgebra $\mathfrak{p}(n)$ is the Lie subalgebra of $\mathfrak{gl}(n+1, n+1)$ whose elements are matrices of the form

$$(3.1.1) \quad M = \left(\begin{array}{c|c} A & B \\ \hline C & -A^t \end{array} \right),$$

where $A \in \mathfrak{sl}_{n+1}$, $B = B^t$ and $C^t = -C$. Throughout this section, we will denote $\mathfrak{p}(n)$ by \mathfrak{g} .

The even part, \mathfrak{g}_0 , is isomorphic to the Lie algebra \mathfrak{sl}_{n+1} . The structure of \mathfrak{g}_1 is the following. Let $S^2(\mathbb{C}^{n+1})$ (resp. $\Lambda^2(\mathbb{C}^{n+1})^*$) denote the second symmetric (resp. exterior) power of \mathbb{C}^{n+1} (resp. $(\mathbb{C}^{n+1})^*$), with the action of \mathfrak{sl}_{n+1} induced by matrix multiplication, and let \mathfrak{g}_1^+ (resp. \mathfrak{g}_1^-) be the set of all matrices of the form (3.1.1) such that $A = C = 0$ (resp. $A = B = 0$). As a \mathfrak{g}_0 -module, we have: $\mathfrak{g}_1 \cong \mathfrak{g}_1^+ \oplus \mathfrak{g}_1^-$, $\mathfrak{g}_1^+ \cong S^2(\mathbb{C}^{n+1})$ and $\mathfrak{g}_1^- \cong \Lambda^2(\mathbb{C}^{n+1})^*$.

If we set $\mathfrak{g}_{-1} = \mathfrak{g}_1^-$, $\mathfrak{g}_0 = \mathfrak{g}_0$ and $\mathfrak{g}_1 = \mathfrak{g}_1^+$, then $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a \mathbb{Z} -grading of \mathfrak{g} which is compatible with the \mathbb{Z}_2 -grading; that is, $\mathfrak{g}_0 = \mathfrak{g}_0$ and $\mathfrak{g}_1 = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$. Let $\mathfrak{h} \subseteq \mathfrak{g}_0$ be a Cartan subalgebra of \mathfrak{g}_0 . Since \mathfrak{g}_0 is isomorphic to \mathfrak{sl}_{n+1} , we can identify \mathfrak{h} with its dual via the bilinear, nondegenerate, \mathfrak{g}_0 -invariant form $(A_1, A_2) = \text{tr}(A_1 A_2)$. If $\{\varepsilon_1, \dots, \varepsilon_n\}$ is the standard orthogonal basis of \mathfrak{h} , then all the roots of \mathfrak{g} are described as follows:

- Roots of \mathfrak{g}_{-1} : $-\varepsilon_i - \varepsilon_j$, where $1 \leq i < j \leq n$.

- Roots of \mathfrak{g}_0 : $\varepsilon_i - \varepsilon_j$, where $i \neq j$ and $1 \leq i, j \leq n$.
- Roots of \mathfrak{g}_1 : $\varepsilon_i + \varepsilon_j$, where $1 \leq i \leq j \leq n$.

Choose a triangular decomposition $\mathfrak{n}_0^- \oplus \mathfrak{h} \oplus \mathfrak{n}_0^+$ of the Lie algebra \mathfrak{g}_0 such that the positive roots are: $\varepsilon_i - \varepsilon_j$, with $i < j$. We fix a triangular decomposition $\mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ of $\mathfrak{p}(n)$, where $\mathfrak{n}^\pm = \mathfrak{g}_{\pm 1} \oplus \mathfrak{n}_0^\pm$, and denote by $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ a Borel subalgebra of $\mathfrak{p}(n)$. Notice that all the roots of \mathfrak{g}_1 are positive.

3.2. Construction of finite-dimensional irreducible modules. Recall that in this section we are assuming that $\mathfrak{g} = \mathfrak{p}(n)$, $n \geq 2$, and A is an associative and commutative algebra with unit. In this subsection we will construct some finite-dimensional irreducible $\mathfrak{p}(n) \otimes A$ -modules which will be used in the next subsection.

For each $\psi \in (\mathfrak{h} \otimes A)^*$, denote by \mathbb{C}_ψ the one-dimensional $\mathfrak{b} \otimes A$ -module where the action of $\mathfrak{n}^+ \otimes A$ is trivial and the action of $\mathfrak{h} \otimes A$ is given by ψ . Define a Verma type module $M(\psi)$ to be

$$M(\psi) = U(\mathfrak{g} \otimes A) \otimes_{U(\mathfrak{b} \otimes A)} \mathbb{C}_\psi.$$

Since a submodule of $M(\psi)$ is proper if and only if its intersection with \mathbb{C}_ψ is trivial, $M(\psi)$ admits a unique maximal non-proper submodule. Let $V(\psi)$ denote the unique irreducible quotient of $M(\psi)$ by such a submodule.

Proposition 3.1. *Every finite-dimensional irreducible $\mathfrak{g} \otimes A$ -module is isomorphic to $V(\psi)$, for some $\psi \in (\mathfrak{h} \otimes A)^*$.*

Proof. The proof of [Sav14, Proposition 4.5] only requires the existence of a nonzero weight vector $v \in V$. Since $\mathfrak{h} \otimes A$ is abelian, such a vector always exists. \square

The annihilator $\text{Ann}_A(V)$ of a $\mathfrak{g} \otimes A$ -module V is, by definition, the sum of all ideals I of A such that $(\mathfrak{g} \otimes I)V = 0$. The support of V is defined to be

$$\text{Supp}(V) = \text{Supp}(\text{Ann}_A(V)).$$

Proposition 3.2. *The tensor product of two irreducible finite-dimensional $\mathfrak{g} \otimes A$ -modules with disjoint supports is irreducible as well.*

Proof. Assume that V_1, V_2 are irreducible finite-dimensional $\mathfrak{g} \otimes A$ -modules with disjoint supports, and let ρ_1 and ρ_2 be the representations associated to V_1 and V_2 , respectively. If I_1 and I_2 denote their supports, then the representation $\rho_1 \otimes \rho_2$ associated to the action of $\mathfrak{g} \otimes A$ on $V_1 \otimes V_2$ factors through the composition

$$(3.2.1) \quad \mathfrak{g} \otimes A \hookrightarrow (\mathfrak{g} \otimes A) \oplus (\mathfrak{g} \otimes A) \twoheadrightarrow (\mathfrak{g} \otimes A/I_1) \oplus (\mathfrak{g} \otimes A/I_2),$$

where the injective homomorphism on the left is the diagonal map and the surjective homomorphism on the right is induced by the quotient maps. Notice that for $i = 1, 2$, V_i is an irreducible finite-dimensional $\mathfrak{g} \otimes A/I_i$ -module, and its highest weight space has dimension one. Since homomorphisms of modules preserve weight spaces, we have $\text{End}_{\mathfrak{g} \otimes A}(V_1) \cong \text{End}_{\mathfrak{g} \otimes A}(V_2) \cong \mathbb{C}$. Therefore, it follows from [Che95, Proposition 8.4] that $V_1 \otimes V_2$ is an irreducible $(\mathfrak{g} \otimes A/I_1) \oplus (\mathfrak{g} \otimes A/I_2)$ -module.

Now, by the fact that the supports of I_1 and I_2 are disjoint, we have that $I_1 \cap I_2 = I_1 I_2$ and then $A = I_1 + I_2$. Therefore $\mathfrak{g} \otimes (A/I_1 I_2) \cong (\mathfrak{g} \otimes A/I_1) \oplus (\mathfrak{g} \otimes A/I_2)$, and we have the following

commutative diagram:

$$\begin{array}{ccc} \mathfrak{g} \otimes A & \xrightarrow{\Delta} & (\mathfrak{g} \otimes A) \oplus (\mathfrak{g} \otimes A) \\ \downarrow & & \downarrow \\ \mathfrak{g} \otimes A/I_1 I_2 & \xrightarrow{\cong} & (\mathfrak{g} \otimes A/I_1) \oplus (\mathfrak{g} \otimes A/I_2). \end{array}$$

This proves that (3.2.1) is surjective, and therefore our result follows. \square

Proposition 3.3. *Let $\psi \in (\mathfrak{h} \otimes A)^*$. The weight spaces of $V(\psi)$ are finite-dimensional if and only if there exists an ideal I of A of finite codimension such that $(\mathfrak{g} \otimes I)V(\psi) = 0$.*

Proof. Suppose all the weight spaces of $V(\psi)$ are finite-dimensional and let v be a highest weight vector of $V(\psi)$. Let Δ^- denote the set $\{-\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}$. For each $\alpha \in \Delta^-$, define I_α to be $\{a \in A \mid (\mathfrak{g}_\alpha \otimes a)v = 0\}$. Since each weight space of $V(\psi)$ is finite-dimensional, I_α is an ideal of A of finite codimension. Let I be $\bigcap_{\alpha \in \Delta^-} I_\alpha$. Since \mathfrak{g} is finite-dimensional, I is an intersection of finitely many finite-codimensional ideals. In particular, I is also a finite-codimensional ideal of A . We claim that $(\mathfrak{g} \otimes I)V(\psi) = 0$. Indeed, the fact that $(\mathfrak{n}^+ \otimes A)v = 0$ follows from the fact that v is a highest weight vector. The fact that $(\mathfrak{n}^- \otimes I)v = 0$ follows from the construction of I . Finally, notice that $\mathfrak{h} \subseteq [\mathfrak{n}^-, \mathfrak{n}^+]$, and so $(\mathfrak{h} \otimes I)v \subseteq [\mathfrak{n}^- \otimes I, \mathfrak{n}^+ \otimes A]v = 0$. Thus $(\mathfrak{g} \otimes I)v = 0$ and hence $W = \{w \in V(\psi) \mid (\mathfrak{g} \otimes I)v = 0\}$ is a non-trivial submodule of $V(\psi)$. Since $V(\psi)$ is irreducible, we conclude that $W = V(\psi)$.

Suppose now there exists a finite-codimensional ideal I of A such that $(\mathfrak{g} \otimes I)V(\psi) = 0$. Then the action of $\mathfrak{g} \otimes A$ on $V(\psi)$ factors through an action of the finite-dimensional Lie superalgebra $\mathfrak{g} \otimes A/I$. Thus, by standard arguments using the PBW Theorem, all the weight spaces of $V(\psi)$ are finite-dimensional. \square

For the remainder of this section, we assume that A is finitely generated.

Proposition 3.4. *Let $\psi \in (\mathfrak{h} \otimes A)^*$. The support of $V(\psi)$ is finite if and only if there exists a finite-codimensional ideal I of A such that $(\mathfrak{g} \otimes I)V(\psi) = 0$.*

Proof. Recall that $\text{Supp } V(\psi) = \text{Supp } \text{Ann}_A(V(\psi))$. So $\text{Ann}_A(V(\psi))$ has finite codimension if and only if there exists a finite-codimensional ideal I of A such that $(\mathfrak{g} \otimes I)V(\psi) = 0$. Since A is finitely generated, $\text{Ann}_A(V(\psi))$ has finite codimension if and only if its support is finite. \square

Before stating the next result, recall that \mathfrak{g}_0 is a finite-dimensional simple Lie algebra.

Proposition 3.5. *If V is an irreducible finite-dimensional $\mathfrak{g} \otimes A$ -module, then $(\mathfrak{g} \otimes J)V = 0$ for some radical ideal J of A of finite codimension.*

Proof. Since V is irreducible and has finite dimension, Proposition 3.3 implies that $(\mathfrak{g} \otimes I)V = 0$ for some finite-codimensional ideal I of A . Let $J = \sqrt{I}$ be the radical of I . We will show that $(\mathfrak{g} \otimes J)V = 0$. Since we are assuming that A is finitely generated (and in particular, Noetherian), there exists some power of J that is contained in I . Then $\mathfrak{g} \otimes (I/J)$ is a solvable Lie superalgebra satisfying the following property:

$$[(\mathfrak{g} \otimes (J/I))_{\bar{1}}, (\mathfrak{g} \otimes (J/I))_{\bar{1}}] = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \otimes (J^2/I) \subseteq \mathfrak{g}_0 \otimes (J^2/I) = [(\mathfrak{g} \otimes (J/I))_{\bar{0}}, (\mathfrak{g} \otimes (J/I))_{\bar{0}}],$$

where the last equality follows from the fact that \mathfrak{g}_0 is a simple Lie algebra. Therefore, it follows from [Kac77, Proposition 5.2.4], that any irreducible finite-dimensional representation of $\mathfrak{g} \otimes J$ is one-dimensional. Then there exists a nonzero vector $w \in V$, and an element $\mu \in (\mathfrak{g} \otimes J)^*$, such

that $xw = \mu(x)w$ for all $x \in \mathfrak{g} \otimes J$. We claim that $\mu = 0$. Since V is finite-dimensional, for any $z \in \mathfrak{n}^\pm \otimes J$, there exists $m \geq 0$ such that $z^m w = \mu(z)^m w = 0$. In other words, $\mu(\mathfrak{n}^\pm \otimes J) = 0$, and hence $(\mathfrak{n}^\pm \otimes J)w = 0$. Let μ' denote the restriction of μ to $\mathfrak{g}_0 \otimes J$. Since \mathfrak{g}_0 is a simple Lie algebra, it follows that the kernel of μ' must be $\mathfrak{g}_0 \otimes J$. In particular, $\mu'(\mathfrak{h} \otimes J) = 0$, since $\mathfrak{h} \subseteq \mathfrak{g}_0$. We have thus proved that $(\mathfrak{g} \otimes J)w = 0$. Now the result follows from the irreducibility of V along with the fact that $W = \{v \in V \mid (\mathfrak{g} \otimes J)v = 0\}$ is a nonzero submodule of V . \square

3.3. Classification of finite-dimensional irreducible modules. Recall that in this section we are assuming that $\mathfrak{g} = \mathfrak{p}(n)$, $n \geq 2$, and also assume that A is an associative, commutative, finitely generated algebra with unit. In this subsection we will classify all the finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$ -modules.

Recall from Section 2.3 that for every $\mathfrak{m} \in \text{MaxSpec}(A)$ one may consider the composition

$$\text{ev}_{\mathfrak{m}}: \mathfrak{g} \otimes A \rightarrow \mathfrak{g} \otimes A/\mathfrak{g} \otimes \mathfrak{m} \xrightarrow{\cong} \mathfrak{g}.$$

Furthermore for a \mathfrak{g} -module V with associated representation $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, the module $\text{ev}_{\mathfrak{m}}^*(V)$ is defined to be the $\mathfrak{g} \otimes A$ -module with associated representation given by the pull-back of ρ along $\text{ev}_{\mathfrak{m}}$:

$$\text{ev}_{\mathfrak{m}}^*(\rho): \mathfrak{g} \otimes A \xrightarrow{\text{ev}_{\mathfrak{m}}} \mathfrak{g} \xrightarrow{\rho} \mathfrak{gl}(V).$$

The $\mathfrak{g} \otimes A$ -module $\text{ev}_{\mathfrak{m}}^*(V)$ is called an evaluation module and its associated representation $\text{ev}_{\mathfrak{m}}^*(\rho)$ is called an evaluation representation. Define $\text{ev}_{\mathfrak{m}}^{\Gamma*}(\rho)$ to be the restriction of $\text{ev}_{\mathfrak{m}}^*(\rho)$ to $(\mathfrak{g} \otimes A)^\Gamma$.

Theorem 3.6. *Every irreducible finite-dimensional $\mathfrak{g} \otimes A$ -module V is isomorphic to a tensor product of evaluation modules.*

Proof. Since V is finite-dimensional, by Proposition 3.5, there exists a radical ideal I of A of finite codimension such that $(\mathfrak{g} \otimes I)V = 0$. Since A is finitely generated and I has finite codimension, Lemma 2.5 (b) implies that the support of I is finite. If $\text{Supp}(I) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\} \subset \text{MaxSpec}(A)$, then $I = \sqrt{I} = \mathfrak{m}_1 \cdots \mathfrak{m}_n$. Therefore the action of $\mathfrak{g} \otimes A$ on V factors through the map

$$(3.3.1) \quad \mathfrak{g} \otimes A \rightarrow \mathfrak{g} \otimes A/I \xrightarrow{\cong} \bigoplus_{i=1}^n \mathfrak{g} \otimes A/\mathfrak{m}_i \xrightarrow{\cong} \mathfrak{g}^{\oplus n}.$$

In other words, V is isomorphic to the pull-back of an irreducible finite-dimensional $\mathfrak{g}^{\oplus n}$ -module W along (3.3.1). Notice that W is also an irreducible finite-dimensional module for $U(\mathfrak{g}^{\oplus n}) \cong U(\mathfrak{g})^{\otimes n}$. By [Che95, Proposition 8.4], there exist irreducible finite-dimensional modules V_1, \dots, V_n for $U(\mathfrak{g})$ such that W is either isomorphic to $V_1 \otimes \cdots \otimes V_n$ or to a proper submodule of $V_1 \otimes \cdots \otimes V_n$. Since $V_1 \otimes \cdots \otimes V_n$ is irreducible by Proposition 3.2, $W \cong V_1 \otimes \cdots \otimes V_n$, and V is isomorphic to a tensor product of evaluation modules. \square

From now on we will assume that Γ is a finite abelian group acting on \mathfrak{g} and A by automorphisms and such that the action of Γ on $\text{MaxSpec}(A)$ is free.

Let $\text{Irred}(\mathfrak{g})$ (resp. $\text{Irred}(\mathfrak{g} \otimes A)^\Gamma$) be the set of isomorphism classes of irreducible finite-dimensional modules for \mathfrak{g} (resp. $(\mathfrak{g} \otimes A)^\Gamma$). Let $[V] \in \text{Irred}(\mathfrak{g})$ denote the isomorphism class of a \mathfrak{g} -module V . Notice that, if V and V' are isomorphic \mathfrak{g} -modules, then $\text{ev}_{\mathfrak{m}}^{\Gamma*}(V)$ and $\text{ev}_{\mathfrak{m}}^{\Gamma*}(V')$ are isomorphic $(\mathfrak{g} \otimes A)^\Gamma$ -modules. Therefore, for each $[V] \in \text{Irred}(\mathfrak{g})$, we define $\text{ev}_{\mathfrak{m}}^{\Gamma*}[V]$ to be the isomorphism class of $\text{ev}_{\mathfrak{m}}^{\Gamma*}(V)$ in $\text{Irred}(\mathfrak{g} \otimes A)^\Gamma$.

Also recall that the action of Γ on \mathfrak{g} induces an action of Γ on $\text{Irred}(\mathfrak{g})$. Namely, if V is a \mathfrak{g} -module representative of $[V] \in \text{Irred}(\mathfrak{g})$ with associated representation $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, then $\gamma[V] = [V^\gamma]$,

where V^γ is a \mathfrak{g} -module with underlying vector space V and associated representation $\rho': \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ given by $\rho'(x) = \rho(\gamma^{-1}x)$ for all $x \in \mathfrak{g}$.

Let \mathcal{P} be the set of Γ -equivariant functions $\pi: \text{MaxSpec}(A) \rightarrow \text{Irred}(\mathfrak{g})$ such that $\pi(\mathfrak{m}) = [\mathbb{C}]$ for all but finitely many distinct $\mathfrak{m} \in \text{MaxSpec}(A)$. Given $\pi \in \mathcal{P}$, recall that its support is defined to be $\text{Supp}(\pi) = \{\mathfrak{m} \in \text{MaxSpec}(A) \mid \pi(\mathfrak{m}) \text{ is nontrivial}\}$. Let X_* be denote the set of all finite subsets $M \subseteq \text{MaxSpec}(A)$ satisfying the following property: if \mathfrak{m} and \mathfrak{m}' are distinct elements in M , then $\mathfrak{m} \notin \Gamma\mathfrak{m}'$. As in Section 2.3, for each $\pi \in \mathcal{P}$, fix an element $\text{Supp}_*(\pi)$ in X_* containing one element of each Γ -orbit in $\text{Supp}(\pi)$, and define $\mathcal{V}(\pi)$ to be the $(\mathfrak{g} \otimes A)^\Gamma$ -module

$$\mathcal{V}(\pi) = \bigotimes_{\mathfrak{m} \in \text{Supp}_*(\pi)} \text{ev}_{\mathfrak{m}}^{\Gamma*} \pi(\mathfrak{m}).$$

Lemma 3.7. *With the above notation, the following hold:*

- (a) *For every $\pi \in \mathcal{P}$, the isomorphism class of $\mathcal{V}(\pi)$ does not depend on the choice of $\text{Supp}_*(\pi)$.*
- (b) *For every $\pi \in \mathcal{P}$, the $(\mathfrak{g} \otimes A)^\Gamma$ -module $\mathcal{V}(\pi)$ is irreducible.*
- (c) *The map $\mathcal{P} \rightarrow \text{Irred}(\mathfrak{g} \otimes A)^\Gamma$ given by $\pi \mapsto \mathcal{V}(\pi)$ is injective.*

Proof. Part (a) follows from [Sav14, Lemma 5.9]. Part (b) follows from Proposition 3.2 along with the fact that the map $\text{ev}_{\mathfrak{m}}^{\Gamma*}$ is surjective for all $\mathfrak{m} \in \text{MaxSpec}(A)$. Part (c) follows from [Sav14, Proposition 5.11]. Notice that the condition of \mathfrak{g} being basic is not used in the proofs of the results cited from [Sav14]. \square

Proposition 3.8. *Every finite-dimensional $(\mathfrak{g} \otimes A)^\Gamma$ -module V is isomorphic to the restriction of a finite-dimensional $\mathfrak{g} \otimes A$ -module V' whose support is in X_* . Moreover, V is irreducible if and only if V' is.*

Proof. The proof of this fact for any finite-dimensional simple Lie superalgebra is the same as the proof of [Sav14, Proposition 8.5]. Notice that, although the fact that $\text{Supp}(V')$ is an element of X_* is not stated, it is also proved there. \square

Theorem 3.9. *Let A be an associative, commutative, finitely generated algebra with unit, Γ be a finite abelian group acting on A and \mathfrak{g} by automorphisms, and such that the action of Γ on $\text{MaxSpec}(A)$ is free. The map $\mathcal{P} \rightarrow \text{Irred}(\mathfrak{g} \otimes A)^\Gamma$ given by $\pi \mapsto \mathcal{V}(\pi)$ is a bijection.*

Proof. Recall from Lemma 3.7 (c) that the map $\mathcal{P} \rightarrow \text{Irred}(\mathfrak{g} \otimes A)^\Gamma$ given by $\pi \mapsto \mathcal{V}(\pi)$ is injective. Let V be an irreducible finite-dimensional $(\mathfrak{g} \otimes A)^\Gamma$ -module. By Proposition 3.8, V is isomorphic to the restriction of an irreducible finite-dimensional $\mathfrak{g} \otimes A$ -module V' , whose support is in X_* . Hence, by Theorem 3.6, $V' \cong \bigotimes_{i=1}^n \text{ev}_{\mathfrak{m}_i}^*(V_i)$ for some $n \geq 0$, $\{\mathfrak{m}_1, \dots, \mathfrak{m}_n\} \in X_*$ and irreducible finite-dimensional \mathfrak{g} -modules V_1, \dots, V_n . Thus, V is isomorphic to $\mathcal{V}(\pi)$, where $\pi(\mathfrak{m}_i) = [V_i]$ for all $i = 1, \dots, n$, and $\pi(\mathfrak{m}) = [\mathbb{C}]$ for all $\mathfrak{m} \notin \text{Supp}(V')$. \square

4. EXTENSIONS

Throughout this section, we will assume that \mathfrak{g} is a finite-dimensional simple Lie superalgebra, that A is an associative, commutative, finitely generated algebra with unit, that Γ is a finite abelian group acting on \mathfrak{g} and A by automorphisms, and that the action of Γ on $\text{MaxSpec}(A)$ is free.

4.1. First general reduction. In this subsection we use Lemma 2.9 to reduce the problem of describing p -extensions between finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$ -modules to that of describing $H^p((\mathfrak{g} \otimes A)^\Gamma, M)$ for certain finite-dimensional $(\mathfrak{g} \otimes A)^\Gamma$ -modules M .

Proposition 4.1. *Let $\pi, \pi' \in \mathcal{P}$, denote $\kappa(\pi)$ by k_0 and $\kappa(\pi')$ by k'_0 . Then there exist positive integers $n, \ell, q_1, \dots, q_\ell$, maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_\ell$ of A in distinct Γ -orbits, and, for each $1 \leq i \leq \ell$, $1 \leq j \leq q_i$, a finite-dimensional indecomposable $\mathfrak{g} \otimes A/\mathfrak{m}_i^n$ -module $M_{i,j}$, such that*

$$\begin{aligned} \text{Hom}_{\mathbb{C}}(\mathcal{V}(\pi), \mathcal{V}(\pi'))^{\oplus 2^{k_0+k'_0}} &\cong \bigoplus_{\substack{1 \leq j_i \leq q_i \\ 1 \leq i \leq \ell}} \text{ev}_{\mathfrak{m}_1^n}^{\Gamma^*} M_{1,j_1} \otimes \cdots \otimes \text{ev}_{\mathfrak{m}_\ell^n}^{\Gamma^*} M_{\ell,j_\ell} \quad \text{and} \\ \text{Ext}_{(\mathfrak{g} \otimes A)^\Gamma}^p(\mathcal{V}(\pi), \mathcal{V}(\pi'))^{\oplus 2^{k_0+k'_0}} &\cong \bigoplus_{\substack{1 \leq j_i \leq q_i \\ 1 \leq i \leq \ell}} H^p\left((\mathfrak{g} \otimes A)^\Gamma, \text{ev}_{\mathfrak{m}_1^n}^{\Gamma^*} M_{1,j_1} \otimes \cdots \otimes \text{ev}_{\mathfrak{m}_\ell^n}^{\Gamma^*} M_{\ell,j_\ell}\right) \quad \text{for all } p > 0. \end{aligned}$$

Proof. From Remark 2.4, there exist positive integers $n, \ell \geq 0$, maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_\ell$ of A in distinct Γ -orbits, and finite-dimensional irreducible $\mathfrak{g} \otimes A/\mathfrak{m}_i^n$ -modules V_i and V'_i such that

$$\mathcal{V}(\pi) \cong \widehat{\bigotimes_{i=1}^{\ell} \text{ev}_{\mathfrak{m}_i^n}^{\Gamma^*} V_i} \quad \text{and} \quad \mathcal{V}(\pi') \cong \widehat{\bigotimes_{i=1}^{\ell} \text{ev}_{\mathfrak{m}_i^n}^{\Gamma^*} V'_i}.$$

Since $k_0 = \kappa(\pi)$ and $k'_0 = \kappa(\pi')$, there are isomorphisms of $(\mathfrak{g} \otimes A)^\Gamma$ -modules

$$\begin{aligned} \text{Hom}_{\mathbb{C}}(\mathcal{V}(\pi), \mathcal{V}(\pi'))^{\oplus 2^{k_0+k'_0}} &\cong \text{Hom}_{\mathbb{C}}\left(\mathcal{V}(\pi)^{\oplus 2^{k_0}}, \mathcal{V}(\pi')^{\oplus 2^{k'_0}}\right) \\ &\cong \left(\bigotimes_{i=1}^{\ell} \text{ev}_{\mathfrak{m}_i^n}^{\Gamma^*} V_i^*\right) \otimes \left(\bigotimes_{i=1}^{\ell} \text{ev}_{\mathfrak{m}_i^n}^{\Gamma^*} V'_i\right) \\ &\cong \bigotimes_{i=1}^{\ell} \text{ev}_{\mathfrak{m}_i^n}^{\Gamma^*} (V_i^* \otimes V'_i). \end{aligned}$$

For each $i = 1, \dots, \ell$, since V_i^* and V'_i are finite-dimensional, there exist $q_i > 0$ and indecomposable $\mathfrak{g} \otimes A/\mathfrak{m}_i^n$ -modules, $M_{i,1}, \dots, M_{i,q_i}$, such that $V_i^* \otimes V'_i \cong \bigoplus_{j=1}^{q_i} M_{i,j}$. Thus there exist isomorphisms of $(\mathfrak{g} \otimes A)^\Gamma$ -modules

$$\bigotimes_{i=1}^{\ell} \text{ev}_{\mathfrak{m}_i^n}^{\Gamma^*} (V_i^* \otimes V'_i) \cong \bigotimes_{i=1}^{\ell} \left(\bigoplus_{j_i=1}^{q_i} \text{ev}_{\mathfrak{m}_i^n}^{\Gamma^*} (M_{i,j_i}) \right) \cong \bigoplus_{\substack{1 \leq j_i \leq q_i \\ 1 \leq i \leq \ell}} \text{ev}_{\mathfrak{m}_1^n}^{\Gamma^*} M_{1,j_1} \otimes \cdots \otimes \text{ev}_{\mathfrak{m}_\ell^n}^{\Gamma^*} M_{\ell,j_\ell}.$$

This proves the first statement. The second statement follows from the first one and Lemma 2.9. \square

The next result is a particular case of Proposition 4.1, thus we omit its proof. It will be used in the proof of Proposition 5.1.

Corollary 4.2. *Let $\pi, \pi' \in \mathcal{P}$, n, ℓ be positive integers and $\mathfrak{m}_1, \dots, \mathfrak{m}_\ell$ be maximal ideals of A in distinct Γ -orbits, and finite-dimensional irreducible $\mathfrak{g} \otimes A/\mathfrak{m}_i^n$ -modules V_i and V'_i , such that $\mathcal{V}(\pi) \cong \widehat{\bigotimes_{i=1}^{\ell} \text{ev}_{\mathfrak{m}_i^n}^{\Gamma^*} V_i}$ and $\mathcal{V}(\pi') \cong \widehat{\bigotimes_{i=1}^{\ell} \text{ev}_{\mathfrak{m}_i^n}^{\Gamma^*} V'_i}$. Denote $\kappa(\pi) = k_0$ and $\kappa(\pi') = k'_0$. If, for every $i = 1, \dots, \ell$, $V_i^* \otimes V'_i$ is a completely reducible $\mathfrak{g} \otimes A/\mathfrak{m}_i^n$ -module, then there exist positive integers q_1, \dots, q_ℓ and, for each $1 \leq i \leq \ell$, $1 \leq j_i \leq q_i$, there exists a finite-dimensional irreducible*

$\mathfrak{g} \otimes A/\mathfrak{m}_i^n$ -module L_{i,j_i} , such that

$$\begin{aligned} \mathrm{Hom}_{\mathbb{C}}(\mathcal{V}(\pi), \mathcal{V}(\pi'))^{\oplus 2^{k_0+k'_0}} &\cong \bigoplus_{\substack{1 \leq j_i \leq q_i \\ 1 \leq i \leq \ell}} \left(\widehat{\bigotimes}_{i=1}^{\ell} \mathrm{ev}_{\mathfrak{m}_i^n}^{\Gamma^*} L_{i,j_i} \right)^{\oplus 2^{\kappa(L_{1,j_1}, \dots, L_{\ell,j_{\ell}})}} \quad \text{and} \\ \mathrm{Ext}_{(\mathfrak{g} \otimes A)^{\Gamma}}^p(V, V')^{\oplus 2^{k_0+k'_0}} &\cong \bigoplus_{\substack{1 \leq j_i \leq q_i \\ 1 \leq i \leq \ell}} H^p \left((\mathfrak{g} \otimes A)^{\Gamma}, \widehat{\bigotimes}_{i=1}^{\ell} \mathrm{ev}_{\mathfrak{m}_i^n}^{\Gamma^*} L_{i,j_i} \right)^{\oplus 2^{\kappa(L_{1,j_1}, \dots, L_{\ell,j_{\ell}})}} \quad \text{for all } p > 0. \end{aligned}$$

Notice that Proposition 4.1 (resp. Corollary 4.2) reduces the computation of p -extensions between finite-dimensional irreducible $(\mathfrak{g} \otimes A)^{\Gamma}$ -modules to the computation of the p -cohomology of finite-dimensional indecomposable (resp. irreducible) $(\mathfrak{g} \otimes A)^{\Gamma}$ -modules.

We finish this subsection by recording a particular case of Corollary 4.2 that will be used in the proof of Corollary 4.7.

Corollary 4.3. *Let V and V' be finite-dimensional irreducible $(\mathfrak{g} \otimes A)^{\Gamma}$ -modules. If the supports of V and V' are disjoint, then $V^* \widehat{\otimes} V'$ is irreducible and*

$$\mathrm{Ext}_{(\mathfrak{g} \otimes A)^{\Gamma}}^p(V, V') \cong \begin{cases} H^p((\mathfrak{g} \otimes A)^{\Gamma}, V^* \widehat{\otimes} V'), & \text{if } V^* \otimes V' \text{ is irreducible,} \\ H^p((\mathfrak{g} \otimes A)^{\Gamma}, V^* \widehat{\otimes} V')^{\oplus 2}, & \text{otherwise,} \end{cases}$$

for all $p > 0$. □

4.2. Second general reduction. In this subsection we will reduce the problem of determining 1-extensions between finite-dimensional $(\mathfrak{g} \otimes A)^{\Gamma}$ -modules to that of determining homomorphisms and extensions between finite-dimensional $\mathfrak{g} \otimes A/\mathfrak{m}^n$ -modules, where \mathfrak{m} is a maximal ideal of A and n is a positive integer. We start with a general result regarding first cohomology.

Proposition 4.4. *If M is a finite-dimensional $(\mathfrak{g} \otimes A)^{\Gamma}$ -module, then there exists a finite-codimensional Γ -invariant ideal $I \subset A$, such that*

$$H^1((\mathfrak{g} \otimes A)^{\Gamma}, M) \cong H^1((\mathfrak{g} \otimes A/I)^{\Gamma}, M) \oplus H',$$

where H' is the kernel of the transgression homomorphism

$$t: \mathrm{Hom}_{(\mathfrak{g} \otimes A/I)^{\Gamma}}((\mathfrak{g} \otimes I/I^2)^{\Gamma}, M) \rightarrow H^2((\mathfrak{g} \otimes A/I)^{\Gamma}, M)$$

induced by the differential $d^1: \mathrm{Hom}_{\mathbb{C}}((\mathfrak{g} \otimes A)^{\Gamma}, M) \rightarrow \mathrm{Hom}_{\mathbb{C}}(\Lambda^2(\mathfrak{g} \otimes A)^{\Gamma}, M)$ defined in (2.5.4).

Proof. Since M is a finite-dimensional $(\mathfrak{g} \otimes A)^{\Gamma}$ -module, by Proposition 2.8, there exists a Γ -invariant finite-codimensional ideal $I \subset A$ such that $(\mathfrak{g} \otimes I)^{\Gamma} M = 0$. By Proposition 2.12, there exists a first-quadrant cohomology spectral sequence associated to $(\mathfrak{g} \otimes A)^{\Gamma}$ and $(\mathfrak{g} \otimes I)^{\Gamma}$, namely

$$(4.2.1) \quad E_2^{p,q} \cong H^p((\mathfrak{g} \otimes A/I)^{\Gamma}, H^q((\mathfrak{g} \otimes I)^{\Gamma}, M)) \Rightarrow H^{p+q}((\mathfrak{g} \otimes A)^{\Gamma}, M).$$

Since (4.2.1) is a first-quadrant cohomology spectral sequence, we have an isomorphism of vector spaces $H^1((\mathfrak{g} \otimes A)^{\Gamma}, M) \cong E_{\infty}^{1,0} \oplus E_{\infty}^{0,1}$. Moreover,

$$E_{\infty}^{1,0} = E_2^{1,0} \cong H^1((\mathfrak{g} \otimes A/I)^{\Gamma}, M) \quad \text{and} \quad E_{\infty}^{0,1} = E_3^{0,1} \cong \ker(d_2^{0,1}: E_2^{0,1} \rightarrow E_2^{2,0}),$$

where $d_2^{0,1}$ is induced by the differential $d^1: \mathrm{Hom}_{\mathbb{C}}((\mathfrak{g} \otimes A)^{\Gamma}, M) \rightarrow \mathrm{Hom}_{\mathbb{C}}(\Lambda^2(\mathfrak{g} \otimes A)^{\Gamma}, M)$ and is known as the transgression homomorphism.

In order to finish the proof, we only need to describe $E_2^{0,1}$ and $E_2^{2,0}$. By (4.2.1),

$$E_2^{0,1} \cong H^0((\mathfrak{g} \otimes A/I)^\Gamma, H^1((\mathfrak{g} \otimes I)^\Gamma, M)) \quad \text{and} \quad E_2^{2,0} \cong H^2((\mathfrak{g} \otimes A/I)^\Gamma, M).$$

Since $(\mathfrak{g} \otimes I)^\Gamma$ acts trivially on M , by Lemma 2.10, there is an isomorphism of $(\mathfrak{g} \otimes A/I)^\Gamma$ -modules $H^\bullet((\mathfrak{g} \otimes I)^\Gamma, M) \cong H^\bullet((\mathfrak{g} \otimes I)^\Gamma, \mathbb{C}) \otimes M$. Moreover, by Lemma 2.14, $H^1((\mathfrak{g} \otimes I)^\Gamma, \mathbb{C})$ is isomorphic to $((\mathfrak{g} \otimes I/I^2)^\Gamma)^*$ as a $(\mathfrak{g} \otimes A/I)^\Gamma$ -module. Thus $E_2^{0,1} \cong \text{Hom}_{(\mathfrak{g} \otimes A/I)^\Gamma}((\mathfrak{g} \otimes I/I^2)^\Gamma, M)$. \square

As a consequence of Lemma 2.9 and Proposition 4.4, we obtain the following result.

Corollary 4.5. *If V, V' are finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$ -modules, then $\text{Ext}_{(\mathfrak{g} \otimes A)^\Gamma}^1(V, V')$ is finite-dimensional.*

Proof. Since V and V' are finite-dimensional modules, by Lemma 2.9, $\text{Ext}_{(\mathfrak{g} \otimes A)^\Gamma}^1(V, V')$ is isomorphic to $H^1((\mathfrak{g} \otimes A)^\Gamma, V^* \otimes V')$. By Proposition 4.4, $H^1((\mathfrak{g} \otimes A)^\Gamma, V^* \otimes V')$ is isomorphic to a subspace of

$$(4.2.2) \quad H^1((\mathfrak{g} \otimes A/I)^\Gamma, V^* \otimes V') \oplus \text{Hom}_{(\mathfrak{g} \otimes A/I)^\Gamma}((\mathfrak{g} \otimes I/I^2)^\Gamma, V^* \otimes V'),$$

where I is a finite-codimensional Γ -invariant ideal of A . Since $\mathfrak{g}, A/I, I/I^2$ and $V^* \otimes V'$ are finite-dimensional, both terms in (4.2.2) are finite-dimensional. This proves that $H^1((\mathfrak{g} \otimes A)^\Gamma, V^* \otimes V')$ is finite-dimensional, and finishes the proof. \square

We emphasize the relevance of Corollary 4.5 by contrasting it with a case where Ext^1 is not finite-dimensional. Namely, let Γ be a group acting by automorphisms on an abelian Lie superalgebra \mathfrak{a} and on an associative commutative algebra with unit B . For any finite-dimensional trivial $(\mathfrak{a} \otimes B)^\Gamma$ -modules M and M' ,

$$\text{Ext}_{(\mathfrak{a} \otimes B)^\Gamma}^1(M, M') \cong \text{Hom}_{\mathbb{C}}((\mathfrak{a} \otimes B)^\Gamma, M^* \otimes M')$$

is finite-dimensional if and only if $(\mathfrak{a} \otimes B)^\Gamma$ is finite-dimensional. In particular, when Γ is trivial and B is infinite-dimensional, $\text{Ext}_{(\mathfrak{a} \otimes B)^\Gamma}^1(M, M')$ is infinite-dimensional.

Now, recall from Corollary 2.15 that $H^1((\mathfrak{g} \otimes A)^\Gamma, \mathbb{C}) = 0$. The next result shows a vanishing condition for $H^1((\mathfrak{g} \otimes A)^\Gamma, M)$ when M is a finite-dimensional $(\mathfrak{g} \otimes A)^\Gamma$ -module of the form $\bigotimes_{i=1}^\ell \text{ev}_{\mathfrak{m}_i}^{\Gamma_{n_i}^*} M_i$.

Proposition 4.6. *Let ℓ, n_1, \dots, n_ℓ be positive integers, $\mathfrak{m}_1, \dots, \mathfrak{m}_\ell$ be maximal ideals of A in distinct Γ -orbits, M_1, \dots, M_ℓ be finite-dimensional $\mathfrak{g} \otimes A/\mathfrak{m}_i^{n_i}$ -modules, and $M = \bigotimes_{i=1}^\ell \text{ev}_{\mathfrak{m}_i}^{\Gamma_{n_i}^*} M_i$. If $\text{Hom}_{\mathfrak{g} \otimes A/\mathfrak{m}_i^{n_i}}(\mathbb{C}, M_i) = 0$ for more than one index i , then $H^1((\mathfrak{g} \otimes A)^\Gamma, M) = 0$.*

Proof. By Proposition 4.4, $H^1((\mathfrak{g} \otimes A)^\Gamma, M)$ is a subspace of

$$H^1((\mathfrak{g} \otimes A/I)^\Gamma, M) \oplus \text{Hom}_{(\mathfrak{g} \otimes A/I)^\Gamma}((\mathfrak{g} \otimes I/I^2)^\Gamma, M),$$

with $I = \prod_{\gamma \in \Gamma} (\gamma \mathfrak{m}_1)^{n_1} \cdots \prod_{\gamma \in \Gamma} (\gamma \mathfrak{m}_\ell)^{n_\ell}$. Since the ideals $\mathfrak{m}_i^{n_i}$ and $\mathfrak{m}_j^{n_j}$ are comaximal and are in distinct Γ -orbits for $i \neq j$, by the Chinese Remainder Theorem, there exists an isomorphism of Lie algebras

$$(4.2.3) \quad (\mathfrak{g} \otimes A/I)^\Gamma \cong \bigoplus_{i=1}^\ell \mathfrak{g} \otimes A/\mathfrak{m}_i^{n_i},$$

and by the Chinese Remainder Theorem for Modules ([DF04, Exercise 10.3.17]), there exists an isomorphism of $(\mathfrak{g} \otimes A/I)^\Gamma$ -modules

$$(4.2.4) \quad (\mathfrak{g} \otimes I/I^2)^\Gamma \cong \bigoplus_{i=1}^{\ell} (\mathfrak{g} \otimes I/\mathfrak{m}_i^{n_i} I).$$

where $(\mathfrak{g} \otimes I/\mathfrak{m}_i^{n_i} I)$ is a trivial module for $\mathfrak{g} \otimes A/\mathfrak{m}_j^{n_j}$ if $j \neq i$. Using isomorphisms (4.2.3) and (4.2.4) and Proposition 2.11, we obtain

$$\begin{aligned} H^1((\mathfrak{g} \otimes A/I)^\Gamma, M) &\cong \bigoplus_{i=1}^{\ell} \left(H^1(\mathfrak{g} \otimes A/\mathfrak{m}_i^{n_i}, M_i) \otimes \bigotimes_{j \neq i} \text{Hom}_{\mathfrak{g} \otimes A/\mathfrak{m}_j^{n_j}}(\mathbb{C}, M_j) \right) \quad \text{and} \\ \text{Hom}_{(\mathfrak{g} \otimes A/I)^\Gamma}((\mathfrak{g} \otimes I/I^2)^\Gamma, M) &\cong \text{Hom}_{\mathfrak{g} \otimes A/\mathfrak{m}_i^{n_i}}(\mathfrak{g} \otimes I/\mathfrak{m}_i^{n_i} I, M_i) \otimes \bigotimes_{j \neq i} \text{Hom}_{\mathfrak{g} \otimes A/\mathfrak{m}_j^{n_j}}(\mathbb{C}, M_j). \end{aligned}$$

Thus, if $\text{Hom}_{\mathfrak{g} \otimes A/\mathfrak{m}_k^{n_k}}(\mathbb{C}, M_k) = 0$ for more than one k , then for each i , there will exist $j \neq i$ such that $\text{Hom}_{\mathfrak{g} \otimes A/\mathfrak{m}_j^{n_j}}(\mathbb{C}, M_j) = 0$. This shows that

$$H^1((\mathfrak{g} \otimes A/I)^\Gamma, M) = \text{Hom}_{(\mathfrak{g} \otimes A/I)^\Gamma}((\mathfrak{g} \otimes I/I^2)^\Gamma, M) = 0. \quad \square$$

The next result generalizes [Kod10, Lemma 3.3] and [NS15, Proposition 3.6]. It follows directly from Corollary 4.3 (for $p = 1$), the classification of finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$ -modules and Proposition 4.6, thus we omit its proof.

Corollary 4.7. *Let V and V' be nontrivial, finite-dimensional, irreducible $(\mathfrak{g} \otimes A)^\Gamma$ -modules. If the supports of V and V' are disjoint, then $\text{Ext}_{(\mathfrak{g} \otimes A)^\Gamma}^1(V, V') = 0$. \square*

We finish this section with the main result of this paper. It describes 1-extensions between finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$ -modules in terms of homomorphisms and extensions between finite-dimensional $\mathfrak{g} \otimes A/\mathfrak{m}^n$ -modules.

Theorem 4.8. *Let \mathfrak{g} be a finite-dimensional simple Lie superalgebra, A be an associative, commutative, finitely generated algebra with unit, Γ be a finite abelian group acting on \mathfrak{g} and A by automorphisms, such that the action of Γ on $\text{MaxSpec}(A)$ is free, and let $\pi, \pi' \in \mathcal{P}$. Let ℓ, n be positive integers, $\mathfrak{m}_1, \dots, \mathfrak{m}_\ell$ be maximal ideals of A in distinct Γ -orbits, and V_i, V'_i be finite-dimensional irreducible $\mathfrak{g} \otimes A/\mathfrak{m}_i^n$ -modules such that $\mathcal{V}(\pi) = \widehat{\bigotimes}_{i=1}^{\ell} \text{ev}_{\mathfrak{m}_i^n}^{\Gamma} V_i$ and $\mathcal{V}(\pi') = \widehat{\bigotimes}_{i=1}^{\ell} \text{ev}_{\mathfrak{m}_i^n}^{\Gamma} V'_i$. For each $i = 1, \dots, \ell$, denote by K_i the kernel of the transgression homomorphism*

$$\text{Hom}_{\mathfrak{g} \otimes A/\mathfrak{m}_i^n}((\mathfrak{g} \otimes I/\mathfrak{m}_i^n I) \otimes V_i, V'_i) \rightarrow \text{Ext}_{(\mathfrak{g} \otimes A/I)^\Gamma}^2(\mathcal{V}(\pi), \mathcal{V}(\pi'))^{\oplus 2\kappa(\pi) + \kappa(\pi')}.$$

- (a) *If V_i is not isomorphic to V'_i for two or more indices i , then $\text{Ext}_{(\mathfrak{g} \otimes A)^\Gamma}^1(\mathcal{V}(\pi), \mathcal{V}(\pi')) = 0$.*
- (b) *If V_i is isomorphic to V'_i for all but one index i , then*

$$\text{Ext}_{(\mathfrak{g} \otimes A)^\Gamma}^1(\mathcal{V}(\pi), \mathcal{V}(\pi'))^{\oplus 2\kappa(\pi) + \kappa(\pi')} \cong \text{Ext}_{\mathfrak{g} \otimes A/\mathfrak{m}_i^n}^1(V_i, V'_i) \oplus K_i.$$

- (c) *If V_i is isomorphic to V'_i for all $i = 1, \dots, \ell$, then*

$$\text{Ext}_{(\mathfrak{g} \otimes A)^\Gamma}^1(\mathcal{V}(\pi), \mathcal{V}(\pi'))^{\oplus 2\kappa(\pi) + \kappa(\pi')} \cong \bigoplus_{i=1}^{\ell} \left(\text{Ext}_{\mathfrak{g} \otimes A/\mathfrak{m}_i^n}^1(V_i, V'_i) \oplus K_i \right).$$

Proof. Denote $\left(\bigotimes_{i=1}^{\ell} \text{ev}_{\mathfrak{m}_i^n}^{\Gamma*} V_i\right)$ by V , $\left(\bigotimes_{i=1}^{\ell} \text{ev}_{\mathfrak{m}_i^n}^{\Gamma*} V'_i\right)$ by V' , and recall from Section 2.3 that $V \cong \mathcal{V}(\pi)^{\oplus 2^{\kappa(\pi)}}$ and $V' \cong \mathcal{V}(\pi')^{\oplus 2^{\kappa(\pi')}}$. Notice that $V^* \otimes V'$ is finite-dimensional and that

$$I = \prod_{\gamma \in \Gamma} (\gamma \mathfrak{m}_1)^n \cdots \prod_{\gamma \in \Gamma} (\gamma \mathfrak{m}_{\ell})^n$$

is a finite-codimensional Γ -invariant ideal of A such that $(\mathfrak{g} \otimes I)^{\Gamma}(V^* \otimes V') = 0$. Thus, by Lemma 2.9 and Proposition 4.4,

$$(4.2.5) \quad \text{Ext}_{(\mathfrak{g} \otimes A)^{\Gamma}}^1(\mathcal{V}(\pi), \mathcal{V}(\pi'))^{\oplus 2^{\kappa(\pi) + \kappa(\pi')}} \cong H^1((\mathfrak{g} \otimes A/I)^{\Gamma}, V^* \otimes V') \oplus H',$$

where H' is the kernel of the transgression homomorphism

$$t: \text{Hom}_{(\mathfrak{g} \otimes A/I)^{\Gamma}}((\mathfrak{g} \otimes I/I^2)^{\Gamma}, V^* \otimes V') \rightarrow H^2((\mathfrak{g} \otimes A/I)^{\Gamma}, V^* \otimes V').$$

Now, notice that $V^* \otimes V' \cong \bigotimes_{i=1}^{\ell} \text{ev}_{\mathfrak{m}_i^n}^{\Gamma*}(V_i^* \otimes V'_i)$ and that the ideals \mathfrak{m}_i^n and \mathfrak{m}_j^n are comaximal and in distinct Γ -orbits for $i \neq j$. Thus, by the Chinese Remainder Theorem, there exists an isomorphism of Lie algebras

$$(4.2.6) \quad (\mathfrak{g} \otimes A/I)^{\Gamma} \cong \bigoplus_{i=1}^{\ell} \mathfrak{g} \otimes A/\mathfrak{m}_i^n,$$

and by the Chinese Remainder Theorem for Modules ([DF04, Exercise 10.3.17]), there exists an isomorphism of $(\mathfrak{g} \otimes A/I)^{\Gamma}$ -modules

$$(4.2.7) \quad (\mathfrak{g} \otimes I/I^2)^{\Gamma} \cong \bigoplus_{i=1}^{\ell} (\mathfrak{g} \otimes I/\mathfrak{m}_i^n I).$$

where $(\mathfrak{g} \otimes I/\mathfrak{m}_i^n I)$ is a trivial $\mathfrak{g} \otimes A/\mathfrak{m}_j^n$ -module for $j \neq i$. Using isomorphisms (4.2.6) and (4.2.7), Lemma 2.9, and Proposition 2.11, we obtain

$$(4.2.8) \quad H^1((\mathfrak{g} \otimes A/I)^{\Gamma}, V^* \otimes V') \cong \bigoplus_{i=1}^{\ell} \left(\text{Ext}_{\mathfrak{g} \otimes I/\mathfrak{m}_i^n I}^1(V_i, V'_i) \otimes \bigotimes_{j \neq i} \text{Hom}_{\mathfrak{g} \otimes A/\mathfrak{m}_j^n}(V_j, V'_j) \right) \quad \text{and}$$

$$\text{Hom}_{(\mathfrak{g} \otimes A/I)^{\Gamma}}((\mathfrak{g} \otimes I/I^2)^{\Gamma}, V^* \otimes V') \cong \text{Hom}_{\mathfrak{g} \otimes A/\mathfrak{m}_i^n}((\mathfrak{g} \otimes I/\mathfrak{m}_i^n I) \otimes V_i, V'_i) \otimes \bigotimes_{j \neq i} \text{Hom}_{\mathfrak{g} \otimes A/\mathfrak{m}_j^n}(V_j, V'_j).$$

Notice that, since V_i and V'_i are irreducible $\mathfrak{g} \otimes A/\mathfrak{m}_i^n$ -modules, then

$$\text{Hom}_{\mathfrak{g} \otimes A/\mathfrak{m}_i^n}(V_i, V'_i) \cong \begin{cases} 0, & \text{if } V_i \not\cong V'_i, \\ \mathbb{C}, & \text{if } V_i \cong V'_i. \end{cases}$$

In order to finish the proof, we will analyze each case separately.

- (a) If V_k is not isomorphic to V'_k for two or more indices k , then for each i , there exists $j \neq i$, such that $\text{Hom}_{\mathfrak{g} \otimes A/\mathfrak{m}_i^n}(V_i, V'_i) = 0$. Using (4.2.8), this implies that $H^1((\mathfrak{g} \otimes A/I)^{\Gamma}, V^* \otimes V') = 0$ and $\text{Hom}_{(\mathfrak{g} \otimes A/I)^{\Gamma}}(\mathfrak{g} \otimes I/\mathfrak{m}_i^n I, V^* \otimes V') = 0$. Thus, by (4.2.5), we conclude that

$$\text{Ext}_{(\mathfrak{g} \otimes A)^{\Gamma}}^1(\mathcal{V}(\pi), \mathcal{V}(\pi')) = 0,$$

proving part (a).

(b) If V_i is isomorphic to V'_i for all but one index i , then $\text{Hom}_{\mathfrak{g} \otimes A/\mathfrak{m}_j^n}(V_j, V'_j) = 0$ for all $j \neq i$.

Applying that to (4.2.8), we conclude that

$$\begin{aligned} H^1((\mathfrak{g} \otimes A/I)^\Gamma, V^* \otimes V') &\cong \text{Ext}_{\mathfrak{g} \otimes I/\mathfrak{m}_i^n}^1(V_i, V'_i) \quad \text{and} \\ \text{Hom}_{(\mathfrak{g} \otimes A/I)^\Gamma}(\mathfrak{g} \otimes I/\mathfrak{m}_j^n, V^* \otimes V') &\cong \begin{cases} \text{Hom}_{\mathfrak{g} \otimes A/\mathfrak{m}_i^n}((\mathfrak{g} \otimes I/\mathfrak{m}_i^n) \otimes V_i, V'_i), & \text{if } j = i, \\ 0, & \text{if } j \neq i. \end{cases} \end{aligned}$$

Together with (4.2.5) this proves part (b).

(c) If V_i is isomorphic to V'_i for all indices i , then $\text{Hom}_{\mathfrak{g} \otimes A/\mathfrak{m}_i^n}(V_i, V'_i) \cong \mathbb{C}$ for all $i = 1, \dots, \ell$.

Applying that to (4.2.8), we conclude that

$$\begin{aligned} H^1((\mathfrak{g} \otimes A/I)^\Gamma, V^* \otimes V') &\cong \bigoplus_{i=1}^{\ell} \text{Ext}_{\mathfrak{g} \otimes I/\mathfrak{m}_i^n}^1(V_i, V'_i), \quad \text{and} \\ \text{Hom}_{(\mathfrak{g} \otimes A/I)^\Gamma}(\mathfrak{g} \otimes I/\mathfrak{m}_i^n, V^* \otimes V') &\cong \text{Hom}_{\mathfrak{g} \otimes A/\mathfrak{m}_i^n}((\mathfrak{g} \otimes I/\mathfrak{m}_i^n) \otimes V_i, V'_i) \quad \forall i = 1, \dots, \ell. \end{aligned}$$

Together with (4.2.5), this shows part (c) and finishes the proof of the theorem. \square

5. CASE $B(0, n)$: AN EXPLICIT EXAMPLE

In this section we assume that \mathfrak{g} is a finite-dimensional simple Lie superalgebra, that A is an associative, commutative, finitely generated algebra with unit, and that Γ is trivial. Using techniques developed in Section 4, we are able to describe the block decomposition of the category of finite-dimensional $\mathfrak{g} \otimes A$ -modules in terms of spectral characters when \mathfrak{g} is of type $B(0, n)$.

5.1. Extensions. We begin with a general result, which will be used in the proof of Proposition 5.2. It describes $H^1(\mathfrak{g} \otimes A, V)$ in the case where \mathfrak{g} is of type II, \mathfrak{p} , \mathfrak{q} , H , S or \tilde{S} , and V is a finite-dimensional irreducible $\mathfrak{g} \otimes A$ -module.

Proposition 5.1. *Let \mathfrak{g} be a finite-dimensional simple Lie superalgebra of type II, \mathfrak{p} , \mathfrak{q} , H , S or \tilde{S} , and let V be a finite-dimensional irreducible $\mathfrak{g} \otimes A$ -module. Let ℓ be a positive integer, $\mathfrak{m}_1, \dots, \mathfrak{m}_\ell$ be distinct maximal ideals of A , and V_1, \dots, V_ℓ be finite-dimensional nontrivial irreducible \mathfrak{g} -modules, such that $V \cong \widehat{\bigotimes}_{i=1}^{\ell} \text{ev}_{\mathfrak{m}_i}^* V_i$. If $\ell = 1$, then*

$$H^1(\mathfrak{g} \otimes A, V) \cong H^1(\mathfrak{g}, V_1) \oplus \text{Hom}_{\mathfrak{g}}(\mathfrak{g}, V_1)^{\oplus d},$$

with $d = \dim_{A/\mathfrak{m}_1} \mathfrak{m}_1/\mathfrak{m}_1^2$. If $\ell > 1$, then $H^1(\mathfrak{g} \otimes A, V) = 0$.

Proof. The case $\ell > 1$ follows directly from Proposition 4.6, as V_1, \dots, V_ℓ are assumed to be nontrivial and irreducible. To prove the case $\ell = 1$, first recall from Proposition 2.12 that there exists a first-quadrant cohomology spectral sequence associated to $\mathfrak{g} \otimes A$ and $\mathfrak{g} \otimes \mathfrak{m}_1$, namely

$$E_2^{p,q} \cong H^p(\mathfrak{g} \otimes A/\mathfrak{m}_1, H^q(\mathfrak{g} \otimes \mathfrak{m}_1, V)) \Rightarrow H^{p+q}(\mathfrak{g} \otimes A, V).$$

Then notice that the evaluation map $\text{ev}_{\mathfrak{m}_1}$ splits via the inclusion $i: \mathfrak{g} \rightarrow \mathfrak{g} \otimes A$, $i(x) = x \otimes 1$. Since $\text{ev}_{\mathfrak{m}_1} \circ i = \text{id}$, the maps induced on cohomology, namely the restriction i^* and the inflation $\text{ev}_{\mathfrak{m}_1}^*$, must satisfy $i^* \circ \text{ev}_{\mathfrak{m}_1}^* = \text{id}$. In particular, this implies that $\text{ev}_{\mathfrak{m}_1}^*: H^\bullet(\mathfrak{g}, V) \rightarrow H^\bullet(\mathfrak{g} \otimes A, V)$ is injective and that the transgression map $d_2^{0,1}: E_2^{0,1} \rightarrow E_2^{2,0}$ must vanish. As a consequence,

$$H^1(\mathfrak{g} \otimes A, V) \cong E_\infty^{1,0} \oplus E_\infty^{0,1} \cong E_2^{1,0} \oplus E_2^{0,1} \cong H^1(\mathfrak{g}, V_1) \oplus \text{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \mathfrak{m}_1/\mathfrak{m}_1^2, V_1).$$

To finish the proof, notice that as a \mathfrak{g} -module $\mathfrak{g} \otimes \mathfrak{m}_1/\mathfrak{m}_1^2$ is isomorphic to $\mathfrak{g}^{\oplus d}$. \square

From now on we will assume that $\mathfrak{g} \cong \mathfrak{osp}(1, 2n)$ and $n \geq 0$. Recall from Table 1 that $\mathfrak{osp}(1, 2n)$ is a basic classical superalgebra of type II, and from Section 2.3 that every finite-dimensional irreducible $\mathfrak{g} \otimes A$ -module is isomorphic to an evaluation module. Moreover, by Theorem 2.3, every finite-dimensional \mathfrak{g} -module is completely reducible. As a consequence, $\text{Ext}_{\mathfrak{g}}^1(M, N) = 0$ for any finite-dimensional \mathfrak{g} -modules M and N . Therefore, given any two finite-dimensional $\mathfrak{g} \otimes A$ -modules V and V' , the tensor product $V^* \otimes V'$ is a completely reducible $\mathfrak{g} \otimes A$ -module. We will use these remarks and Theorem 4.8 to compute $\text{Ext}_{\mathfrak{g} \otimes A}^1(V, V')$.

Proposition 5.2. *Let $\mathfrak{g} = \mathfrak{osp}(1, 2n)$, $n \geq 0$, V and V' be finite-dimensional irreducible $\mathfrak{g} \otimes A$ -modules. Let ℓ be a positive integer, $\mathfrak{m}_1, \dots, \mathfrak{m}_\ell$ be distinct maximal ideals of A , and $\lambda_1, \mu_1, \dots, \lambda_\ell, \mu_\ell$ be elements in Λ^+ such that $V = \bigotimes_{i=1}^\ell \text{ev}_{\mathfrak{m}_i}^* V(\lambda_i)$ and $V' = \bigotimes_{i=1}^\ell \text{ev}_{\mathfrak{m}_i}^* V(\mu_i)$. For each $i = 1, \dots, \ell$, denote $\dim_{A/\mathfrak{m}_i} \mathfrak{m}_i/\mathfrak{m}_i^2$ by d_i .*

- (a) *If λ_i and μ_i differ for two or more indices i , then $\text{Ext}_{(\mathfrak{g} \otimes A)^\Gamma}^1(V, V') = 0$.*
 (b) *If λ_i and μ_i differ for precisely one index i , then*

$$\text{Ext}_{\mathfrak{g} \otimes A}^1(V, V') \cong \text{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\lambda_i), V(\mu_i))^{\oplus d_i}.$$

- (c) *If $\lambda_i = \mu_i$ for all $i = 1, \dots, \ell$, then*

$$\text{Ext}_{\mathfrak{g} \otimes A}^1(V, V') \cong \bigoplus_{i=1}^\ell \text{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\lambda_i), V(\mu_i))^{\oplus d_i}.$$

Proof. Part (a) follows directly from Theorem 4.8(a). To prove parts (b) and (c), we will proceed as in the proofs of Theorem 4.8(b) and Theorem 4.8(c) respectively.

First, notice that $V^* \otimes V' = \bigotimes_{i=1}^\ell \text{ev}_{\mathfrak{m}_i}^*(V(\lambda_i^*) \otimes V(\mu_i))$. Since every finite-dimensional \mathfrak{g} -module is completely reducible, there exist positive integers q_1, \dots, q_ℓ and elements $\nu_{i,j} \in \Lambda^+$ such that $V(\lambda_i^*) \otimes V(\mu_i) = \bigoplus_{j=1}^{q_i} V(\nu_{i,j})$ for each $i = 1, \dots, \ell$. Thus

$$\begin{aligned} V^* \otimes V' &\cong \bigoplus_{\substack{1 \leq i \leq \ell \\ 1 \leq j_i \leq q_i}} \text{ev}_{\mathfrak{m}_1}^* V(\nu_{1,j_1}) \otimes \cdots \otimes \text{ev}_{\mathfrak{m}_\ell}^* V(\nu_{\ell,j_\ell}) \quad \text{and} \\ \text{Ext}_{\mathfrak{g} \otimes A}^1(V, V') &\cong \bigoplus_{\substack{1 \leq i \leq \ell \\ 1 \leq j_i \leq q_i}} H^1(\mathfrak{g} \otimes A, \text{ev}_{\mathfrak{m}_1}^* V(\nu_{1,j_1}) \otimes \cdots \otimes \text{ev}_{\mathfrak{m}_\ell}^* V(\nu_{\ell,j_\ell})). \end{aligned}$$

For each $J = (j_1, \dots, j_\ell)$, denote the $\mathfrak{g} \otimes A$ -module $\text{ev}_{\mathfrak{m}_1}^* V(\nu_{1,j_1}) \otimes \cdots \otimes \text{ev}_{\mathfrak{m}_\ell}^* V(\nu_{\ell,j_\ell})$ by L_J . By Corollary 2.15 and Proposition 4.6, $H^1(\mathfrak{g} \otimes A, L_J)$ is nonzero only if ν_{i,j_i} are zero for all but one index i ; that is, $L_J = \text{ev}_{\mathfrak{m}_i}^* V(\nu_{i,j_i})$ for some $i = 1, \dots, \ell$. In this case, as \mathfrak{g} is of type II, Proposition 5.1 gives

$$H^1(\mathfrak{g} \otimes A, L_J) \cong H^1(\mathfrak{g}, V(\nu_{i,j_i})) \oplus \text{Hom}_{\mathfrak{g}}(\mathfrak{g}, V(\nu_{i,j_i}))^{\oplus d_i}.$$

Moreover, since every finite-dimensional \mathfrak{g} -module is completely reducible, $H^1(\mathfrak{g}, V(\nu_{i,j_i})) = 0$ for all $\nu_{i,j_i} \in \Lambda^+$. Now, to finish the proof, observe that

$$\text{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\lambda_k), V(\mu_k)) \cong \text{Hom}_{\mathfrak{g}}(\mathfrak{g}, V(\lambda_k^*) \otimes V(\mu_k)) \cong \bigoplus_{j=1}^{q_k} \text{Hom}_{\mathfrak{g}}(\mathfrak{g}, V(\nu_{k,j}))$$

for all $k = 1, \dots, \ell$, and that ν_{i,j_i} can only be zero if

$$\text{Hom}_{\mathfrak{g}}(V(\lambda_i), V(\mu_i)) \cong \text{Hom}_{\mathfrak{g}}(\mathbb{C}, V(\lambda_i^*) \otimes V(\mu_i)) \cong \bigoplus_{j=1}^{q_i} \text{Hom}_{\mathfrak{g}}(\mathbb{C}, V(\nu_{i,j}))$$

is one-dimensional; that is, only if $\lambda_i = \mu_i$. □

5.2. Block decomposition. Let $\mathfrak{g} = \mathfrak{osp}(1, 2n)$, $n \geq 0$, and let \mathcal{F} denote the abelian category consisting of finite-dimensional $\mathfrak{g} \otimes A$ -modules and their $\mathfrak{g} \otimes A$ -module homomorphisms. Recall from Section 2.3 that, since $\mathfrak{osp}(1, 2n)$ is of type II, every finite-dimensional irreducible $\mathfrak{g} \otimes A$ -module is isomorphic to an evaluation module. Two finite-dimensional irreducible $\mathfrak{g} \otimes A$ -modules V_1 and V_2 are said to be linked when either $V_1 \cong V_2$ or $\text{Ext}_{\mathfrak{g} \otimes A}^1(V_1, V_2) \neq 0$. Consider the smallest equivalence relation on the set of isomorphism classes of finite-dimensional irreducible $\mathfrak{g} \otimes A$ -modules under which two classes are equivalent if their representatives are linked. Denote by \mathcal{B} the set of equivalence classes.

Since every object M of \mathcal{F} is finite-dimensional, by the Jordan-Hölder Theorem, it admits a composition series; that is, a finite chain of submodules $0 = M_0 \subset M_1 \subset \dots \subset M_n = M$, such that M_i/M_{i-1} is irreducible for all $i = 1, \dots, n$. Moreover, for every object M of \mathcal{F} , the irreducible quotients M_i/M_{i-1} appearing in a composition series of M , as well as their multiplicities, do not depend on the choice of composition series. These irreducible modules M_i/M_{i-1} are called composition factors of M .

For each $b \in \mathcal{B}$ and each object M of \mathcal{F} , denote by M_b the largest submodule of M whose composition factors are all in b . A proof of the following result can be obtained by adapting the arguments contained in [Jan03, §II.7.1].

Proposition 5.3. *Let M and M' be objects of \mathcal{F} . Then*

- (a) $M = \bigoplus_{b \in \mathcal{B}} M_b$.
- (b) $\text{Hom}_{\mathfrak{g} \otimes A}(M, M') \cong \bigoplus_{b \in \mathcal{B}} \text{Hom}_{\mathfrak{g} \otimes A}(M_b, M'_b)$.
- (c) $\text{Ext}_{\mathfrak{g} \otimes A}^i(M, M') \cong \bigoplus_{b \in \mathcal{B}} \text{Ext}_{\mathfrak{g} \otimes A}^i(M_b, M'_b)$ for all $i > 0$.

From the first two items of Proposition 5.3, it follows that for each $b \in \mathcal{B}$, the subcategory consisting of the object M of \mathcal{F} such that $M_b = M$ and their homomorphisms is a full subcategory of \mathcal{F} , which we denote by \mathcal{F}_b . Moreover, $\mathcal{F} = \bigoplus_{b \in \mathcal{B}} \mathcal{F}_b$. This is called a block decomposition of the category \mathcal{F} , and \mathcal{F}_b is said to be a block of the category \mathcal{F} .

In order to describe the blocks of \mathcal{F} , we will introduce the notion of spectral characters (compare it with [Kod10, Definition 4.2] and [NS15, Definition 5.9]). First, recall from Section 2.2 that a finite-dimensional irreducible \mathfrak{g} -module is uniquely determined by its highest-weight in Λ^+ , so for each $\mathfrak{m} \in \text{MaxSpec}(A)$, denote by $\pi^{\mathfrak{m}}$ the weight in Λ^+ such that $\pi(\mathfrak{m}) \cong V(\pi^{\mathfrak{m}})$.

Recall from Section 2.2 the definitions of the subgroups Q and Λ of \mathfrak{h}^* .

Definition 5.4. Given a finite-dimensional irreducible $\mathfrak{g} \otimes A$ -module $\mathcal{V}(\pi)$, $\pi \in \mathcal{P}$, define its spectral character to be the function $\chi_\pi: \text{MaxSpec}(A) \rightarrow \Lambda/Q$ given by $\chi_\pi(\mathfrak{m}) = \pi^{\mathfrak{m}} + Q$ for all $\mathfrak{m} \in \text{MaxSpec}(A)$. Given $\pi \in \mathcal{P}$, an object M of \mathcal{F} is said to have spectral character χ_π if every composition factor of M has spectral character χ_π .

In the last result of this paper we describe the block decomposition of the category \mathcal{F} in terms of spectral characters for \mathfrak{g} (compare it with [Kod10, Proposition 4.5] and [NS15, Lemma 5.10]).

Theorem 5.5. *Let $\mathfrak{g} = \mathfrak{osp}(1, 2n)$ and $n \geq 0$. Two finite-dimensional $\mathfrak{g} \otimes A$ -modules V and V' are in the same block, if and only if they have the same spectral character.*

Proof. By Definition 5.4, it is sufficient to prove the statement for distinct, finite-dimensional irreducible $\mathfrak{g} \otimes A$ -modules. Thus, assume that $V = \mathcal{V}(\pi)$, that $V' = \mathcal{V}(\pi')$, and that $\pi \neq \pi' \in \mathcal{P}$.

Also by Definition 5.4, V and V' belong to the same block if and only if there exist distinct finite-dimensional irreducible modules $\mathcal{V}(\pi_0), \mathcal{V}(\pi_1), \dots, \mathcal{V}(\pi_k)$, such that $\pi = \pi_0$, $\pi_k = \pi'$ and either $\text{Ext}_{\mathfrak{g} \otimes A}^1(\mathcal{V}(\pi_i), \mathcal{V}(\pi_{i+1})) \neq 0$ or $\text{Ext}_{\mathfrak{g} \otimes A}^1(\mathcal{V}(\pi_{i+1}), \mathcal{V}(\pi_i)) \neq 0$ for each $i = 0, \dots, k-1$.

By Proposition 5.2, for each $i = 0, \dots, k-1$, $\text{Ext}_{\mathfrak{g} \otimes A}^1(\mathcal{V}(\pi_i), \mathcal{V}(\pi_{i+1})) \neq 0$ if and only if $\pi_i(\mathfrak{m}) \cong \pi_{i+1}(\mathfrak{m})$ for all but precisely one $\mathfrak{m} \in \text{MaxSpec}(A)$ that satisfies $\text{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \pi_i(\mathfrak{m}), \pi_{i+1}(\mathfrak{m})) \neq 0$. Since $\pi_{i+1}(\mathfrak{m})$ is a finite-dimensional irreducible \mathfrak{g} -module, $V(\pi_{i+1}^{\mathfrak{m}})$, and $\mathfrak{g} \otimes \pi_i(\mathfrak{m})$ are completely reducible, $\text{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \pi_i(\mathfrak{m}), \pi_{i+1}(\mathfrak{m})) \neq 0$ if and only if $V(\pi_{i+1}^{\mathfrak{m}})$ is a composition factor of $\mathfrak{g} \otimes \pi_i(\mathfrak{m})$. Since $\pi_i(\mathfrak{m}) \cong V(\pi_i^{\mathfrak{m}})$ and the set of weights of \mathfrak{g} is Δ , we conclude that $\text{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \pi_i(\mathfrak{m}), \pi_{i+1}(\mathfrak{m})) \neq 0$ if and only if $\pi_{i+1}^{\mathfrak{m}} \in \pi_i^{\mathfrak{m}} + \Delta$.

Using the argument of the previous paragraph for each $i = 0, \dots, k-1$, one concludes that $\mathcal{V}(\pi)$ and $\mathcal{V}(\pi')$ belong to the same block if and only if $\pi(\mathfrak{m}) \cong \pi'(\mathfrak{m})$ for all but finitely many maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_j \in \text{MaxSpec}(A)$ that satisfy $\pi^{\mathfrak{m}_i} - (\pi')^{\mathfrak{m}_i} \in Q$ for all $i = 1, \dots, j$, $j \leq k$. \square

Remark 5.6. Notice that the key ingredient that was used in the proof of Theorem 5.5 was the description of 1-extensions obtained in Proposition 5.2(b). In order to generalize Theorem 5.5 to other simple Lie superalgebras and to equivariant map algebras, one needs to obtain a more precise description of the kernel of the transgression map in Theorem 4.8. The three main obstacles that one faces when trying to obtain this more precise description of the kernel of the transgression map are: first, the fact that finite-dimensional \mathfrak{g} -modules are not necessarily completely reducible when \mathfrak{g} is not of type $B(0, n)$; second, the fact that the inclusion $\mathfrak{g} \hookrightarrow (\mathfrak{g} \otimes A)^{\Gamma}$ does not necessarily admit a Lie superalgebra splitting in the equivariant case; and third, the fact that finite-dimensional irreducible $(\mathfrak{g} \otimes A)^{\Gamma}$ -modules are not necessarily isomorphic to evaluation modules in types I and W.

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